Přednáška 13

Teorie aritmetiky a Gödelovy výsledky o neúplnosti a nerozhodnutelnosti

Hilbert calculus

- The set of axioms has to be decidable,
  axiom schemes:

1. $A \supset (B \supset A)$
2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
3. $(\neg B \supset \neg A) \supset (A \supset B)$
4. $\forall x\, A(x) \supset A(x/t)$ Term t substitutable for x in A
5. $(\forall x\, [A \supset B(x)]) \supset (A \supset \forall x\, B(x))$, x is not free in A

Hilbert calculus

The ***deduction rules*** are of a form:

$A_1,\ldots,A_m$ |– $B_1,\ldots,B_m$

enable us to prove ***theorems*** (*provable formulas*) of the calculus. We say that each B_i is *derived* (inferred) from the set of assumptions $A_1,\ldots,A_m$.

Rule schemas:

MP: **A, A ⊃ B |– B** (modus ponens)

G: **A |– ∀*x* A** (generalization)

A Proof from Assumptions

A (direct) proof of a formula A from assumptions $A_1,\dots,A_m$ is a sequence of formulas (proof steps) $B_1,\dots B_n$ such that:

- $A = B_n$ (the proved formula A is the last step)
- each B_i ($i=1,\dots,n$) is either
  - an ***axiom*** *(logically valid formula)*, or
  - an ***assumption*** A_k ($1 \le k \le m$), a formula valid in a *chosen interpretation I*, or
  - B_i is ***derived*** from the previous B_j ($j=1,\dots,i-1$) using a rule of the calculus.

A formula ***A*** is ***provable from*** $A_1, \dots, A_m$, denoted $A_1,\dots,A_m \vdash A$, if there is a proof of A from $A_1,\dots,A_m$.

The Theorem of Deduction

- Let **A be a *closed*** formula, B any formula. Then:
- **$A_1, A_2, \dots, A_k$ |– A ⊃ B** *if and only if* **$A_1, A_2, \dots, A_k$, A |– B**.
- For $k = 0$: **|– A ⊃ B** *if and only if* **A |– B**.

Remark: The statement

a) ***if* |– A ⊃ B, *then* A |– B**

is valid universally, not only for A being a closed formula (the proof is obvious – modus ponens).

On the other hand, the other statement

b) ***If* A |– B, *then* |– A ⊃ B**

is ***not valid*** for an open formula A (with at least one free variable).

- *Example*: Let A = A(x), B = $\forall x$A(x).

  Then **A(x) |– $\forall x$A(x)** is valid according to the generalisation rule.

  But the formula **A(x) ⊃ $\forall x$A(x)** is generally not logically valid, and therefore not provable in a sound calculus.

Theorem on Soundness (semantic consistence)

- Generalisation rule **A(*x*) |– ∀*x*A(*x*)** is tautology preserving and *truth-in-interpretation preserving*:
- If in *a structure I the formula A(x) is true for any valuation e of x,* $\models_I$ **A(*x*),** then, by definition, it means that $\models_I$ **∀*x*A(*x*)** (is true in the interpretation I).

A Complete Calculus:
if |= A then |– A

- Each logically valid formula is provable in the calculus
- The set of *theorems* = the set of *logically valid formulas*
- ***Sound (semantic consistent) and complete calculus***: **|= A iff |– A**
  - Provability and logical validity coincide in FOPL (1st-order predicate logic)
- ***Hilbert calculus is sound and complete***

Properties of a calculus: deduction rules, consistency

- ***The set of deduction rules*** enables us to perform proofs mechanically, considering just the symbols, abstracting of their semantics. Proving in a calculus is a ***syntactic method***.
- A natural demand is a ***syntactic consistency*** of the calculus.
- A ***calculus is consistent*** iff there is a WFF φ such that φ is not provable (*in an inconsistent calculus everything is provable*).
- This definition is equivalent to the following one: a calculus is consistent iff a formula of the form $A \wedge \neg A$, or $\neg(A \supset A)$, is not provable.
- A calculus is syntactically consistent iff it is sound (semantically consistent).

Sound and Complete Calculus: |= A iff |– A

- ***Soundness***
  (an outline of the proof has been done)
- *In 1928* ***Hilbert and Ackermann*** *published a concise small book Grundzüge der theoretischen Logik, in which they arrived at exactly this point: they had defined axioms and derivation rules of predicate logic (slightly distinct from the above), and formulated the problem of completeness. They raised a question whether such a proof calculus is complete in the sense that each logical truth is provable within the calculus; in other words, whether the calculus proves exactly all the logically valid FOPL formulas.*
- ***Completeness Proof (T a set of sentences = closed formulas)***
- ***Stronger version: if T |= φ, then T |– φ. Kurt Gödel, 1930***
- A ***theory T is consistent*** iff there is a formula φ which is not provable in T: $T \nvdash \varphi$.

Strong Completeness of Hilbert Calculus: *if T |= φ, then T |– φ*

- The proof of the Completeness theorem is based on the following ***Lemma***:

Each consistent theory has a model.

- ***if T |= φ, then T |– φ*** iff
- **if *not* T |– φ, then *not* T |= φ** ⇒
- **{T ∪ ¬φ}** does not prove φ as well
  (¬φ does not contradict T) ⇒
- **{T ∪ ¬φ}** is consistent, it has a model M ⇒
- M is a model of T in which φ is not true ⇒
- φ is not entailed by T: **T |≠ φ**

Properties of a calculus: Hilbert calculus is not decidable

- There is another property of calculi. To illustrate it, let's raise a question: having a formula φ, does the calculus *decide* φ?
- In other words, ***is there an algorithm*** that would answer Yes or No, having φ as input and answering the question whether φ is logically valid or no**?** If there is such an algorithm, then the calculus is ***decidable.***
- If the calculus is complete, then it proves all the logically valid formulas, and the proofs can be described in an algorithmic way.
- However, in case the input formula φ is not logically valid, the algorithm does not have to answer (in a final number of steps).
- Indeed, there are *no decidable 1st order predicate logic calculi, i.e.,* ***the problem of logical validity is not decidable*** in the FOPL.
- (the consequence of Gödel Incompleteness Theorems)

Provable = logically true?
Provable from … = logically entailed by …?

- The relation of ***provability*** **($A_1,\ldots,A_n \vdash A$)** and the relation of ***logical entailment*** **($A_1,\ldots,A_n \models A$)** are ***distinct relations.***
- Similarly, the **set of theorems $\vdash A$** (of a calculus) is generally not identical to **the set of logically valid formulas $\models A$**.
- The former is *syntactic and defined within a calculus,* the latter *independent of a calculus, it is semantic.*
- In a *sound* calculus the set of theorems is a *subset* of the set of logically valid formulas.
- In a *sound and complete* calculus the set of theorems is *identical* with the set of logically valid formulas.

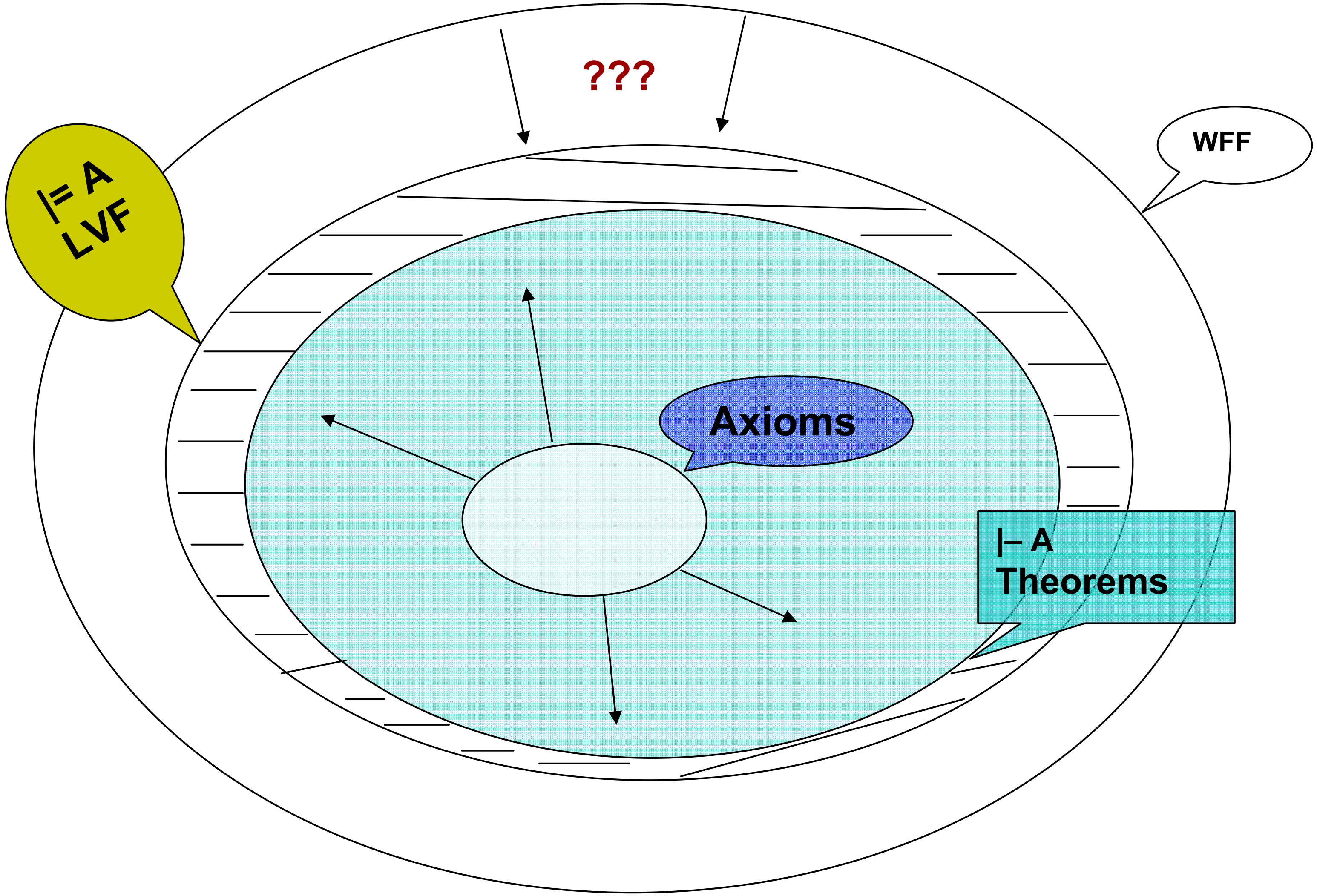
1921: Hilbert's Program of Formalisation of Mathematics

- Kurt Gödel 1929, 1930-doctoral dissertation *Completeness Theorem*;
  $A_1,\ldots,A_n \vdash B$ iff $A_1,\ldots,A_n \models B$
- Continued: 1930 **(!!!)** - Gödel first announced ***Incompleteness Theorem*** to Rudolf Carnap in Café Reichsrat in Vienna.
- The work on incompleteness was published early in 1931, and defended as a *Habilitationschrift* at the University of Vienna in 1932.
- The title of *Privatdozent* gave Gödel the right to give lectures at the university but without pay.

1921: Hilbert's Program of Formalisation of Mathematics

- Reasoning with infinites $\Rightarrow$ paradoxes (Zeno, infinitesimals in the 17^{th} century, Russell, …)
- ***Hilbert***: *'finitary' methods* of axiomatisation and reasoning in mathematics;
- ***Kant***: We obviously cannot experience infinitely many events or move about infinitely far in space.
- However, there is no upper bound on the number of steps we execute, we can always move a step further.
- But at any point we will have acquired only a *finite* amount of experience and have taken only a finite number of steps.
- Thus, for a Kantian like Hilbert, the only legitimate infinity is a ***potential* infinity, not the *actual* infinity**.
- "mathematics is about symbols" (?), mathematical reasoning - Syntactic laws of symbol manipulation (?); ***consistency proof***

Hilbert Calculus: Completeness



Incompleteness of arithmetic, Gödel's first and second theorems

- Now we are not interested just in ***logical truths***, i.e., sentences true under ***every interpretation*** of the FOPL language,
- but in ***sentences characterizing arithmetic of natural numbers*** which are ***true*** under the *standard (intended) interpretation*, which is the structure ***N***:
- $\boldsymbol{N} = \langle \mathbf{N}, \mathbf{0}, \mathbf{S}_{\mathbf{N}}, \mathbf{+}_{\mathbf{N}}, \ast_{\mathbf{N}}, \mathbf{=}_{\mathbf{N}}, \leq_{\mathbf{N}} \rangle$

Theory T: *logical* + *special axioms*, *rules (e.g. of Hilbert)*

- **T |=** φ ⇔ **T |–** φ.
- What is missing? Why did Hilbert want more?
- In order to avoid *inconsistencies (the set of* ***all*** *subsets, …) –* ***proof of consistency*** *of arithmetic*
- *To find a consistent theory whose axioms characterise arithmetic of natural numbers* ***completely****, so that* ***each arithmetic truth*** *expressed in a formal language would be logically entailed by the axioms and thus* ***derivable from*** *them in a* ***finite*** *number of steps.*
- *Moreover, the set of* ***axioms*** *has to be fixed and* ***initially well defined.***
- *Gödel's two theorems on incompleteness show that these demands cannot be met.*

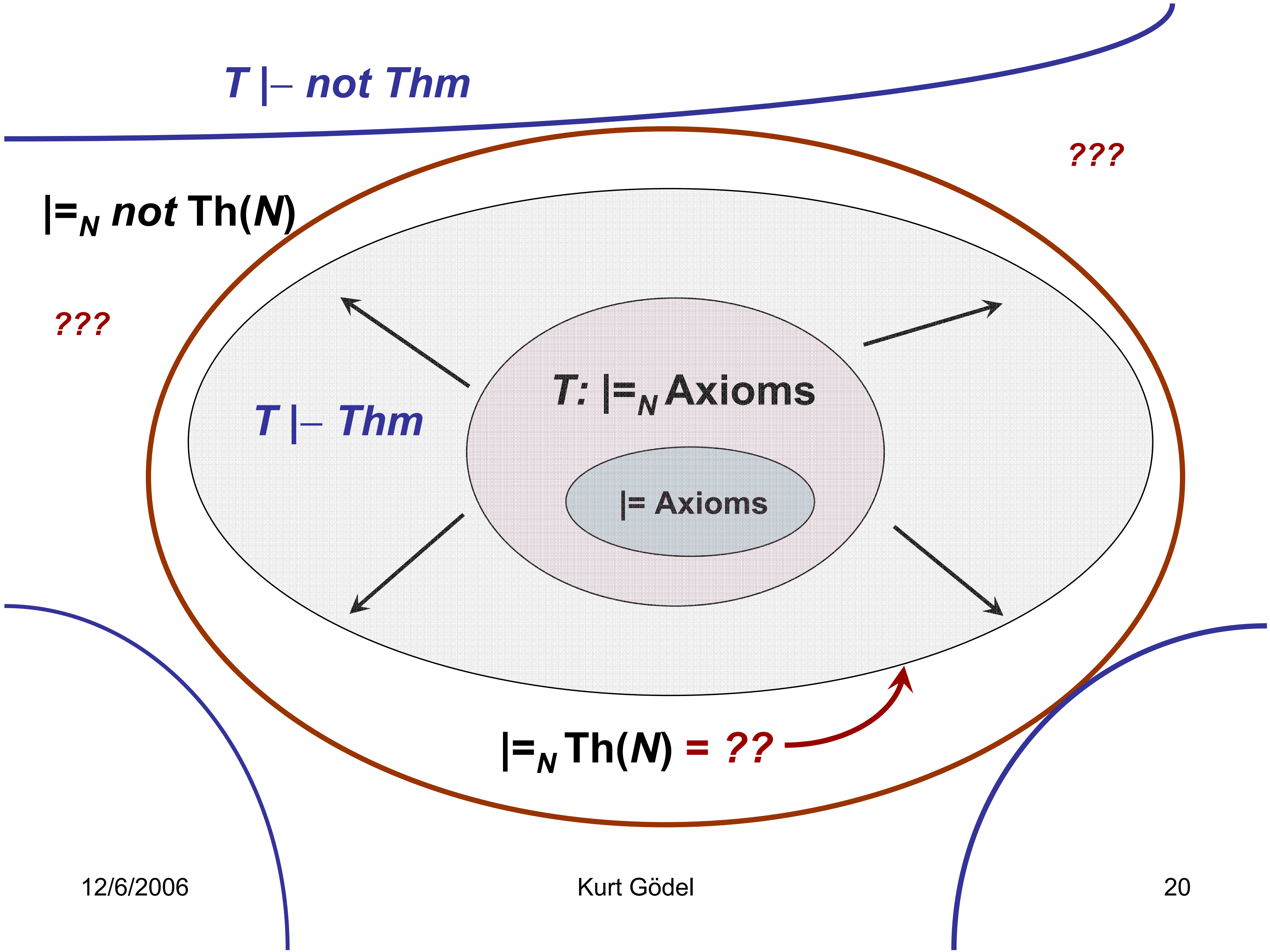
Theory *N: arithmetic*

- constant symbol $\underline{\mathbf{0}}$ (zero)
- unary functional symbol **S** (successor: +1)
- binary functional symbols **+** and $*$ (plus, times)
- binary predicate symbols **=**, $\leq$
- Sentences like:
  - $\forall x \forall y\ (x+y) = (y+x)$ (true in *N*)
  - $\exists x\ (S(S(x)) \leq \underline{0})$ (False in *N*)
- **each sentence φ:**
  - **either $N \models \varphi$, or $N \models \neg\varphi$**

A theory T is complete iff

T is consistent and for each sentence φ it holds that: *T* $\vdash \varphi$ or *T* $\vdash \neg\varphi$;

- in other words, there are no independent sentences;
- *each sentence φ is decidable in T.*
- *Thm(T) – the set of all formulas provable in T:*
  - *THM(T)* = $\{\varphi;\ T \vdash \varphi\}$
- *Th(N) – the set of all sentences True in N – the True arithmetic:*
  - *Th(N)* = $\{\varphi;\ N \models \varphi\}$; **? complete ?**
  - ***THM(T)*** $\subseteq$ ***Th(N); THM(T) = Th(N)***


T |– not Thm
???
|=N not Th(N)
???
T: |=N Axioms
T |– Thm
|= Axioms
|=N Th(N) = ??

Special axioms to characterise arithmetics: a) Robinson (Q)

- $\forall x \quad (Sx \neq \underline{0})$
- $\forall x \forall y \quad (Sx = Sy \supset x = y)$
- $\forall x \quad (x + \underline{0} = x)$
- $\forall x \forall y \quad (x + Sy = S(x + y))$
- $\forall x \quad (x * \underline{0} = \underline{0})$
- $\forall x \forall y \quad (x * Sy = (x * y) + x)$
- $\forall x \forall y \quad (x \leq y = \exists z\, (z + x = y)\,)$

Robinson's theory Q: *N is its model.* Q is a *weak theory*.

It proves only simple sentences like: $5 + 1 = 6$

General simple statements like commutativity of + or $*$, i.e., sentences like

$\forall x \forall y\, (x + y = y + x)$,
$\forall x \forall y\, (x * y = y * x)$,

are not provable in Q.

However, it proves ***all*** *the* ***Σ-sentences*** *that should be provable, i.e., the Σ-sentences* ***true in N: if σ is a Σ-sentence such that N |= σ, then Q |– σ.***

Σ-sentences – syntactically simple

- Syntactical complexity: a number of alternating quantifiers. *(actual infinity !)*
- an arithmetic formula φ is formed from a formula ψ by a *bounded quantification*, if φ has one of the following forms:
- $\forall \boldsymbol{v}\ (\boldsymbol{v} < \boldsymbol{x} \supset \psi)$, $\exists \boldsymbol{v}\ (\boldsymbol{v} < \boldsymbol{x}\ \&\ \psi)$, $\forall \boldsymbol{v}\ (\boldsymbol{v} \leq \boldsymbol{x} \supset \psi)$, $\exists \boldsymbol{v}\ (\boldsymbol{v} \leq \boldsymbol{x}\ \&\ \psi)$, where *v, x* are distinct variables, $\forall \boldsymbol{v}$, $\exists \boldsymbol{v}$ - ***bounded quantifiers***.
- A formula φ is a *bounded formula* if it contains only bounded quantifiers. A formula φ is a ***Σ-formula***, if φ is formed from bounded formulas using only $\wedge$, $\vee$, $\exists$, and any bounded quantifiers.

Peano arithmetic *PA*

- Q is *Σ-complete*;
- *PA arithmetic* = Q + *induction axioms:*
- $[\varphi(\underline{0}) \wedge \forall x\, (\varphi(x) \supset \varphi(Sx))] \supset \forall x\, \varphi(x)$ **!!** *Actual inf.*
- *PA* is "reasonable", it conforms to finitism: we added a "geometrical pattern" of formulas – axiom schema.
- *N |= PA, N is a **standard model**.*
- *Terms denoting numbers:* $\underline{3} = SSS\underline{0}$, … *(numerals)*
- $\underline{2} + \underline{3} = SS\underline{0} + SSS\underline{0} = SSSSS\underline{0} = \underline{5}$

Peano arithmetic *PA is not complete*

- *PA is a **strong** theory and many laws of arithmetic are provable in it;*
- *however, there is a sentence(s) φ:*
- ***N |=** φ but **PA not |–** φ. And, of course,*
- ***PA not |–** $\neg\varphi$ as well, because $\neg\varphi$ **is not true** in **N** and PA proves only sentences true in its models.* (soundness ***assumption***)
- Well, let us add some axioms or rules … ?
- ***No way: you cannot know in advance, which should be added …***

Recursive axiomatisation (finitism!)

- A theory T is ***recursively axiomatized*** if there is an ***algorithm*** that for any formula φ decides whether φ is an axiom of the theory or not.
- Algorithm: a ***finite procedure*** that for any input formula φ gives a "Yes / No" output in a finite number of steps.
- Due to *Church's thesis* it can be explicated by any *computational model*, e.g., *Turing machine*.

PA **is *arithmetically sound:* all arithmetic sentences provable in T are valid in N.**

- ***Gödel's first theorem on incompleteness !***
- Let *T* be a theory that contains *Q* (i.e., the language of *T* contains the language of arithmetic and *T* proves all the axioms of *Q*).
- Let *T* be *recursively axiomatized* and *arithmetically sound*. ($\boldsymbol{T} \vdash \varphi \Rightarrow \boldsymbol{N} \models \varphi$)
- Then *T* is an ***incomplete theory***, i.e., there is a sentence φ independent of *T*:
  - $\varphi \in Th(N)$; $\varphi \notin Thm(T)$
- *T* proves neither φ nor $\neg\varphi$.

What did Goedel prove?

- it is not possible to find a *recursively axiomatized* consistent theory, in which *all the true arithmetic sentences* about natural numbers could be proved.
- ***Either*** you have a (semantically complete) naïve arithmetic = all the sentences true in N – ***not*** recursively axiomatisable
- ***Or*** you have an incomplete theory
- *Completeness* × *recursive axiomatisation*

Summary and outline of the Proof

1. An arithmetic theory such as Peano arithmetic (*PA*) is adequate: it encodes finite sequences of numbers and defines sequence operations such as concatenation (sss(0),+, …).
2. In an adequate theory *T* we can ***encode*** the ***syntax of terms***, ***sentences*** (closed formulas) and ***proofs***.
   - Let us denote the ***code*** (e.g. ASCII) of φ as $\mathbf{<\varphi>}$.
3. ***Self-Reference (diagonal) lemma***: For any formula φ**(*x*)** (with one free variable) there is a sentence ψ such that ψ **iff** $\varphi\mathbf{(<\psi>)}$.
4. Let ***Th(N)*** be the set of numbers that ***encode true sentences of arithmetic*** (i.e. formulas true in the standard model of arithmetic), and ***Thm(T)*** the set of numbers that ***encode sentences provable in an adequate (sound) theory T***.
5. Since the theory is sound, ***Thm(T)*** $\subseteq$ ***Th(N)***.
6. It would be nice if they were the same; in that case the theory T would be complete.

Summary and outline of the Proof

7. No such luck if the theory T is recursively axiomatized, i.e., if the set of axioms is *computable* (algorithm … Yes, No).

- ***Computability*** of the set of axioms ***and completeness*** of the theory T are two goals that ***cannot be met together***, because:
- The set ***Th(N)*** is ***not even definable*** by an arithmetic sentence (that would be true if its number were in the set and false if not):
- Let *n* be a number such that ***n*** $\notin$ ***Th(N)***.
- Then by the Self Reference (3) there is a sentence φ such that **<φ> = *n***.
- Hence φ iff **<φ>** $\notin$ ***Th(N)*** iff *φ* ***is not true in N*** iff **not** φ – ***contradiction***. There is no such φ. (Liar's Paradox)

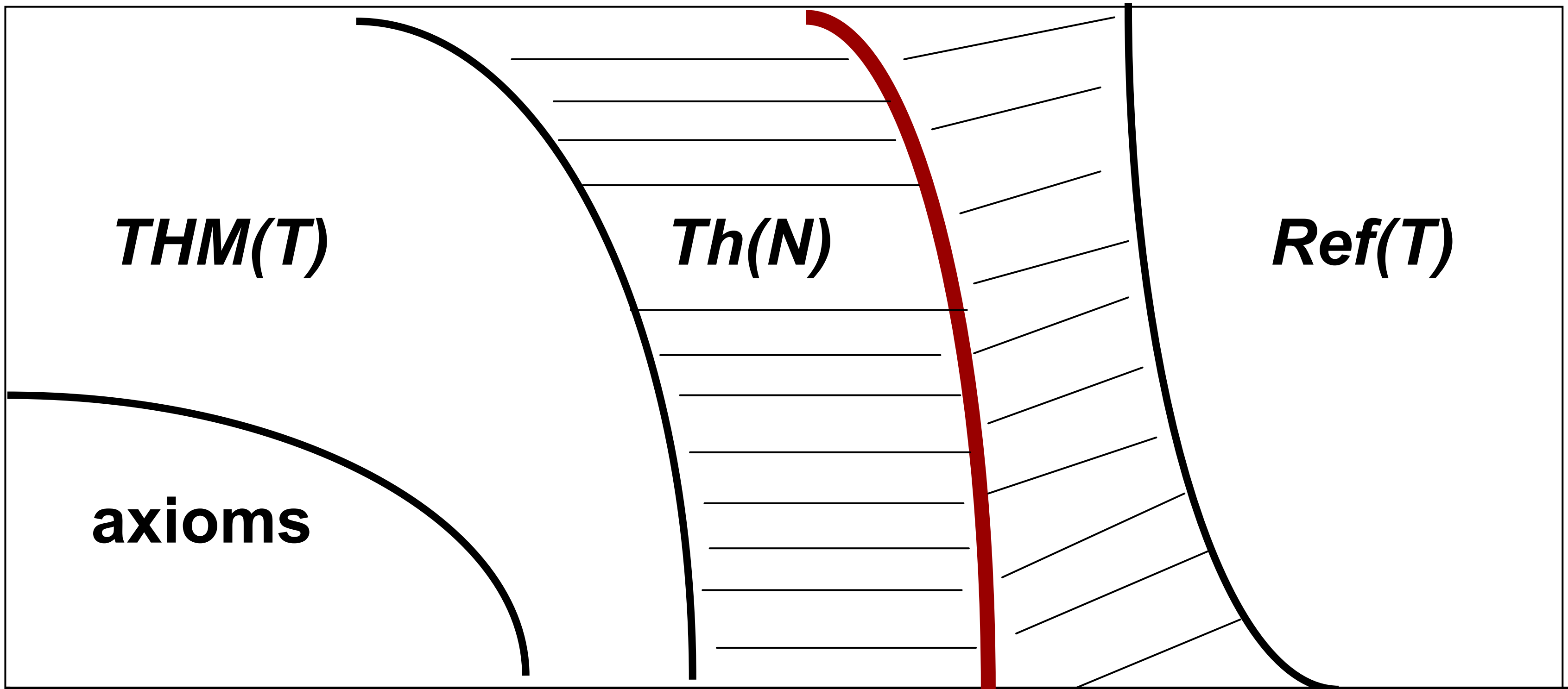
Summary and outline of the Proof

8. Since undefinable implies uncomputable **there will never be a program that would decide whether an arithmetic sentence is true or false (in the standard model of arithmetic)**.
9. The set ***Thm(T)*** is ***definable*** in an adequate theory, say Q:
   - φ**:** the number **<φ>** ∈ ***Thm(T)*** iff ***T* |**– φ, for:
   - the set of axioms is recursively enumerable, i.e., computable,
   - so is the set of proofs that use these axioms and
   - so is the set of provable formulas, ***Thm(T)***.
   - ***Thm(T)* is definable**.
   - Let ***n*** ∉ ***Thm(T)***. By the Self Reference - there is a sentence φ such that **<φ> = *n***.
   - Hence φ **iff <φ>** ∉ ***Thm(T)*** iff φ is **not provable**. This is impossible in a sound theory: provable sentences are true. Hence φ **is true but not provable in *T***.

Decidability

- A *theory T is* ***decidable*** if the set ***Thm(T)*** of formulas provable in T is (generally) recursive (i.e., computable).
- If a theory is recursively axiomatized and complete, then it is decidable.
- consequence of Gödel’s incompleteness theorem:
- ***No*** **recursively axiomatized** theory T that contains Q and has a model N**, is** ***decidable***: there is **no algorithm** that would decide every formula φ (whether it is provable in the theory T or not).

THM(T)-*provable by T;* ***Th(N)***- *true in N;* ***Ref(T)***--*T proves* $\neg\varphi$



If the (consistent) theory ***T*** is recursively axiomatized and complete, then ***Thm(T) = Th(N)***, and ***Ref(T)*** is a complement of them. ***But PA is not.***

Gödelův důkaz detailněji

1. **Kódování**: *efektivní 1-1 zobrazení množiny syntaktických objektů do množiny přirozených čísel (injekce), např. ASCII*
2. ***Teorie rekurzivních funkcí*** (po Gödelovi):
   - (partial) *recursive functions* = algorithmically computable.
   - A set S is *recursively enumerable* if there is a partial recursive function *f* such that S is a domain of *f*: Dom(*f*) = S. („počítá“ S, ale nemusí počítat komplement S)
   - A set S is a *(general) recursive set* if its characteristic function is a (total) recursive function – („počítá S i komplement S“)
3. **Formule definují množiny**: A(*x*) definuje množinu A_S těch prvků *a* universa, pro které $|=_I$ A(*x*)[*e*], *e*(*x*) = *a*

Gödelův důkaz detailněji

4. **<u>Σ-úplnost teorie Q:</u>** Σ-sentence dokazatelné v Q jsou právě všechny pravdivé v *N.*
   - Σ-formulas define just all the algorithmically computable, i.e., recursively enumerable sets of natural numbers.
5. ***Dok(x) je* Σ-formule**, která definuje množinu ***Thm(T)*** – množinu čísel těch formulí, které jsou dokazatelné v *T. Tedy:*
6. ***T |– φ iff <φ> ∈ Thm(T) iff N |= Dok(<φ>)***

Gödelův důkaz detailněji

7. Gödelovo **<u>diagonální lemma</u>**: *For any formula $\psi(x)$ of the arithmetic language with one free variable there is a sentence φ such that $\varphi \equiv \psi\,\mathbf{(}<\varphi>\mathbf{)}$ is provable in Q.* Hence:
8. rovnice $\mathbf{Q} \vdash \varphi \equiv \psi(\underline{<\varphi>})$ – neznámá φ má vždy pro libovolné ψ řešení, a to nezávisle na kódování.

   Metafora: φ říká “Já mám vlastnost ψ”.

Diagonální lemma – netriviální aplikace, volba predikátu ψ

- Aplikace self-reference:
- *Alfred Tarski* (slavný polský logik) aplikace Epimenidova paradoxu lháře („já jsem nepravdivá“): ***neexistuje*** definice pravdivosti pro všechny formule: $N \models \varphi$ iff $N \models True(\underline{<\varphi>})$.
  - Neexistuje formule *True(x), která by definovala množinu Th(N) – kódů formulí pravdivých v N.*
  - $Q \vdash \omega \equiv \neg True(\underline{<\omega>})$, $T \vdash \omega \equiv \neg Tr(\underline{<\omega>})$. *But*
  - $T \vdash \omega \equiv Tr(\underline{<\omega>})$ – *spor.*

Diagonální lemma – netriviální aplikace, volba predikátu ψ

- *Gödel's sentence claims "I am not provable", Rosser's sentence says that "each my proof is preceded by a smaller proof of my negation".*

9. Aplikuj diagonální lemma na $\neg$***Dok(x)*** *!! - žádný paradox, Dok(x) definuje Thm(T)!*
 T |– φ iff <φ> $\in$ Thm(T) iff N |= Dok(<φ>)

10. ***Gödel's diagonal formula ν such that Q |– $\nu \equiv \neg$Dok(<u><ν></u>)). Thus we have:***

- ***ν iff <ν> $\notin$ Thm(T) iff ν is not provable in T.***

Gödelova formule ν *je nezávislá na teorii T a přitom pravdivá v T !*

- Kdyby $T \vdash \nu$ pak by $N \models Dok(<\nu>)$. Ale
- *Dok(<ν>) je* Σ-formule, tedy $T \vdash Dok(<\nu>)$.
- ***Dok(<ν>)*** $\equiv \neg \nu$, tedy $T \vdash \neg \nu$. ***Spor*** *(pokud není T nekonzistentní. Ale to není – má model N).*
- Tedy $N \models \neg Dok(<\nu>)$ a $N \models \nu$, **ale** $T \vdash \neg \nu$.
- ***T je neúplná teorie, nedemonstruje všechny aritmetické pravdy.***

Důsledky

- Žádná rekurzivně axiomatizovaná „rozumná“ aritmetika (obsahující aspoň Q) ***není rozhodnutelná*** (algoritmus by se dal lehko zobecnit na dokazatelnost).
- Problém logické pravdivosti není rozhodnutelný v kalkulu PL1 – v „prázdné teorii“ bez speciálních axiomů.
- *Neexistuje algoritmus, který by rozhodoval dokazatelnost v kalkulu, a tedy logickou pravdivost.*

Alonzo Church: parciální rozhodnutelnost

- Množina *Thm(kalkulu) teorémů kalkulu je rekurzivně spočetná, ale není rekurzivní:*
- *„Dostaneme se“ výpočtem – algoritmem na všechny logicky pravdivé formule, ale nerozhodneme komplement Thm(kalkulu).*
- *Pokud φ je logicky pravdivá, v konečném čase algoritmus (třeba rezoluční metoda) odpoví. Jinak může cyklovat.*

Gödel's Second Theorem on incompleteness.

- In any consistent recursively axiomatizable theory *T* that is strong enough to encode sequences of numbers (and thus the syntactic notions of "formula", "sentence", "proof") the consistency of the theory *T* is **not provable in *T***.
- „Já jsem nedokazatelná" je ekvivalentní „Neexistuje formule φ taková, že <φ> a <¬φ> jsou dokazatelné v *T*".

Proč Hilbert tak nutně potřeboval důkaz konzistence?

- Vždyť *PA* má model *N*! ***Ale***: tento předpoklad množiny *N* přirozených čísel jakožto modelu je předpoklad ***aktuálního nekonečna.***
- *Co když zase „vyskočí“ paradoxy? Víme jak „vypadají“* ***hodně velká přirozená čísla?***
- ***PA má také jiné modely, které nejsou isomorfní s N ! (indukce)***
- $[\varphi(\underline{0}) \wedge \forall x\, (\varphi(x) \supset \varphi(Sx))] \supset \forall x\, \varphi(x)$

Proč Hilbert tak nutně potřeboval důkaz konzistence?

- *Roughly:* ***T* |= φ iff *T* |– φ (strong Completeness).**
- *Now the sentence ν:* ***T* not |– φ, $\Rightarrow$ ν is not valid in every model of *T*.**
- ***But - standard model N |= φ, which is a model of T.***
- ***Every model isomorphic to N is also a model of T;***
- ***ν is however not valid in every model of T. Hence T must have a non-standard model.***

Conclusion

- Gödelovy výsledky změnily tvář moderní matematiky: rozvoj teorie rekurzivních funkcí, computability, computer science, … atd.
- Possible impact of Gödel's results on the philosophy of mind, artificial intelligence, and on Platonism ….
- Gödel himself suggested that the human mind cannot be a machine and that Platonism is correct.
- Most recently *Roger Penrose* has argued that "the Gödel's results show that the whole programme of artificial intelligence is wrong, that creative mathematicians do not think in a mechanic way, but that they often have a kind of insight into the Platonic realm which exists independently from us"