Two Standard and Two Modal Squares of Opposition Studied in Transparent Intensional Logic

Logika: systémový rámec rozvoje oboru v ČR a koncepce logických propedeutik
pro mezioborová studia (reg. č. CZ.1.07/2.2.00/28.0216, OPVK)

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Abstract

In this paper, we examine modern reading of the Square of Opposition by means of intensional logic. Explicit use of possible world semantics helps us to sharply discriminate between the standard and modal ('alethic') readings of categorical statements. We get thus two basic versions of the Square. The Modal Square has not been introduced in the contemporary debate yet and so it is in the heart of interest. It seems that some properties ascribed by mediaeval logicians to the Square require a shift from its Standard to its Modal version. Not necessarily so, because for each of the two there is its mate which can be easily confused with it. The discrimination between the initial and modified versions of the Standard and Modal Square enable us to sharply separate findings about logical properties of the Square into four groups, which makes their proper comparison possible.

Keywords: Square of Opposition; modal Square of Opposition; modality; intensional logic; Math. Subject Classification: 03A05

some terminology:

- the *Standard* Square of Opposition = the Square with categorical statements
- the *Modal* Square of Opposition = the Square with modal versions of categorical statements
- *classical* reading etc. = what is held by classical logicians (followers of Aristotle)
- *modern* reading etc. = what is based on modern logic or held in modern logic textbooks

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I.

A very brief introduction to Transparent Intensional Logic

I.1 A very brief introduction to Transparent Intensional Logic

- *Transparent Intensional Logic* (*TIL*) developed by Pavel Tichý (1936 Brno - 1994 Dunedin, New Zealand) in the very beginning of 1970s
- TIL can be seen as a *typed λ-calculus*, i.e. a higher-order logic (with careful formation of its terms)
- till now, most important applications of TIL are in semantics of natural language (propositional attitudes, modalities, subjunctive conditionals, verb tenses, etc. are analysed in TIL; I will suppress temporal parameter), rivalling thus the system of Montague

I.2 TIL semantic scheme

expression *E*

| *E expresses*

construction *C* (= *meaning* explicated as an *hyperintension*)

| *E denotes*, *C means*

intension/extension (= an PWS-style of explication of *denotation*)

- *constructions* are structured abstract entities of algorithmic nature
- they are written by λ-terms: *constants* | *variables* | *compositions* | *λ-closures*
- 'intensional principle' of individuation: every object *O* is constructed by infinitely many *congruent*, but *not identical* constructions *Cs*
- every construction *C* is thus *specified by*:

  i. the object *O* constructed by *C*, ii. the way how *C* constructs the object *O*

I.3 Type theory of TIL

- Tichý modified Church's Simple Theory of Types (and ramified it in 1988, which is omitted here; the type of *k*-order constructions is $*_k$)
- Let base *B* be a non-empty class of pairwise disjoint collections of atomic objects, e.g. $B_{TIL}=\{\iota,o,\omega,\tau\}$:

  a) Any member of *B* is a *type over B*.

  b) If $\alpha_1, ..., \alpha_m, \beta$ are types over *B*, then $(\beta\alpha_1...\alpha_m)$ – i.e. the collection of all total and partial *m*-ary functions from $\alpha_1, ..., \alpha_m$ to β – is a *type over B*.

- (possible world) *intensions* (*propositions*, *properties*, ...) are functions from possible worlds

I.4 Types of some basic objects

- "/" abbreviates "*v*-constructs an object of type"

- x/ξ (a ξ-object, i.e. an object belonging to the type ξ)
- $p/(o\omega)$ (a proposition); let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be concrete examples of constructions of propositions
- $f, g/((o\xi)\omega)$ (a property of ξ-objects; its *extension* in W is of type $(o\xi)$); let $\boldsymbol{F}$ and $\boldsymbol{G}$ be concrete examples of constructions of properties
- $\forall^{\xi}/(o(o\xi))$ (the class containing the only universal class of ξ-objects; $\forall$={U})
- $\exists^{\xi}/(o(o\xi))$ (the class containing all nonempty classes of ξ-objects)
- **1**, **0**/o (True, False); *o*/o (a truth value); $\neg/(oo)$ (the classical negation); $\wedge, \vee, \rightarrow, \leftrightarrow/(ooo)$ (the classical conjunction, disjunction, material conditional, equivalence); $=^{\xi}/(o\xi\xi)$ (a familiar relation between ξ-objects); $\neq^{\xi}/(o\xi\xi)$; '$^{\xi}$' will be usually suppressed even in the case of other functions/relations

I.5 Definitions

- Tichý's system of deduction for his simple type theory (1976, 1982)
- *sequents* are made from *matches* x:C ("the variable or trivialization x v-constructs the same ξ-object as the compound construction C", loosely: "C=x")
- *definitions* are certain deduction rules of form

$$|\text{-}\ x{:}C \Leftrightarrow x{:}D$$

where C and D are different constructions of the same object as x; $\Leftrightarrow$ means interderivability of sequents flanking the $\Leftrightarrow$ sign
- "$C \Leftrightarrow D$" *abbreviates* "$|\text{-}\ x{:}C \Leftrightarrow x{:}D$"
- example (where $\varnothing$/($o\xi$), the total empty ξ-class):

$$\varnothing \Leftrightarrow_{\mathrm{df}} \lambda x\ \mathrm{F}$$

I.6 Definiteness - overcoming partiality failure

- when adopting partiality, most classical laws do not hold (JR 2008)
- for instance, De Morgan Law for exchange of quantifiers must be amended (JR 2008, 2010) to be protected against the case when the extensions of the properties *f* and *g* are not total classes (which would cause *v*-improperness of $[[f_w\, x] \rightarrow [g_w\, x]]$ and then invalidity of the law):

$$\neg\forall\lambda x.[f_w\, x]! \rightarrow [g_w\, x]! \quad \Leftrightarrow \quad \exists\lambda x.\neg[[f_w\, x]! \rightarrow [g_w\, x]!]$$

(on the right side, ! can be omitted)

- "[...*w*...]!" *abbreviates* "$[\mathbf{True}^{T\pi}{}_w\, \lambda w'[...w'...]]$"

- w/ω (a possible world); $\mathbf{True}^{T\pi}/((o(o\omega))\omega)$ (a property of propositions);

$$[\mathbf{True}^{T\pi}{}_w\, p] \Leftrightarrow_{df} \exists\lambda o.[o{=}p_w] \wedge [o{=}1]$$

(compare it with $[\mathbf{True}^{P\pi}{}_w\, p] \Leftrightarrow_{df} [p_w{=}1]$)

I.7 Properties of objects/constructions

- properties of constructions supervene on properties of objects (e.g. propositions) constructed by those constructions

- for instance, the truth$^{\pi}$ of a proposition P makes all constructions of P true*

- **True**$^{*k\mathrm{PT}}$ $/((o*_k)\omega)$ (a property of k-order constructions);

$$[\mathbf{True}^{*k\mathrm{PT}}{}_w\ c^k] \Leftrightarrow_{\mathrm{df}} [\mathbf{True}^{\pi\mathrm{T}}{}_w\ {}^2c^k]$$

(^2C v-constructs the object, if any, v-constructed by C)

- for another example (c^k,d^k/$*_k$ (a k-order construction)):

$$[p \models^{\pi} q] \quad \Leftrightarrow_{\mathrm{df}} \quad \forall\lambda w[p_w \rightarrow q_w] \qquad \text{(where } \models^{\pi}/(o(o\omega)(o\omega))\text{)}$$

$$[c^k \models d^k] \quad \Leftrightarrow_{\mathrm{df}} \quad [{}^2c^k \models^{\pi} {}^2d^k] \qquad \text{(where } \models/(o*_k*_k)\text{)}$$

II.

Modern reading of the Standard Square

II.1 Modern reading of the Standard Square

- common records of the 4 constructions in vertices are written *red*

"Every *F* is *G*"

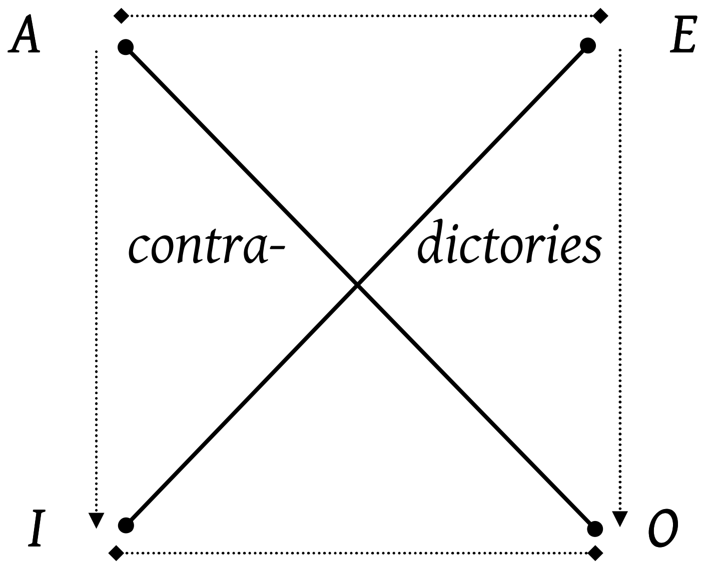
$\lambda w.\forall\lambda x.[F_w\, x]!\rightarrow[G_w\, x]!$

"None *F* is *G*"

$\lambda w.\forall\lambda x.[F_w\, x]!\rightarrow\neg[G_w\, x]!$

$\forall x\ (Fx\rightarrow Gx)$

$\forall x\ (Fx\rightarrow\neg Gx)$



$\exists x\ (Fx\wedge Gx)$

$\exists x\ (Fx\wedge\neg Gx)$

"Some *F* is *G*"

$\lambda w.\exists\lambda x.[F_w\, x]!\wedge[G_w\, x]!$

"Some *F* are not *G*"

$\lambda w.\exists\lambda x.[F_w\, x]!\wedge\neg[G_w\, x]!$

II.2 Standard contradictories and equivalences

- *contradictories* (e.g. "Every *F* is *G*" contradicts "Not every *F* is *G*", i.e. "Some *F* is not *G*") are fully confirmed, i.e. they match with the classical view

$$[\mathbf{Contradictory}\ P\ Q] \quad \Leftrightarrow_{\mathrm{df}} \forall\lambda w.\neg[P_w! \leftrightarrow Q_w!]$$

  (where **Contradictory**/(ο(οω)(οω)), analogously below)
- (remark: usual formalization of (other) relations in the Square (*P* and *Q* are: contraries iff $P\uparrow Q$, subcontraries iff $P\vee Q$, *Q* is subalternate of *P* iff $P\rightarrow Q$) ignores modality, which obfuscates the cause of their invalidity in modal interpretation of the Square; cf. below)
- *contrapositions* (e.g. "Every *F* is *G*" ↔ "Every non-*G* is non-*F*") and *obversions* (e.g. "No *F* is *G*" ↔ "Every *F* is non-*G*") are also fully confirmed (using function Non- (((οι)ω)(οι)ω)), JR 2007: $[[\mathbf{Non\text{-}}\ f]_w\ x] \Leftrightarrow_{\mathrm{df}} \neg[f_w\ x]$)

II.3 The modern reading of the Standard Square and generalized quantifiers

- we can reformulate the 4 categorical statements using generalized quantifiers All, Some and No (Tichý 1976, JR 2009):

$[[\mathrm{All}\ f_w]\ g_w]$	$\Leftrightarrow_{df}$	$\forall\lambda x.[f_w x]!\rightarrow[g_w x]!$	(btw. $\Leftrightarrow [f_w \subseteq g_w]$)
$[[\mathrm{No}\ f_w]\ g_w]$	$\Leftrightarrow_{df}$	$\forall\lambda x.[f_w x]!\rightarrow\neg[g_w x]!$	(btw. $\Leftrightarrow [[f_w \cap g_w]=\varnothing]$)
$[[\mathrm{Some}\ f_w]\ g_w]$	$\Leftrightarrow_{df}$	$\exists\lambda x.[f_w x]!\wedge[g_w x]!$	(btw. $\Leftrightarrow [[f_w \cap g_w]\neq\varnothing]$)

- obviously, we will reach the very same results concerning contradictories, contrapositions and obversions (cf. Tichý 1976)

- remark: realize that $\neg[[\mathrm{All}\ f_w]\ g_w] \Leftrightarrow \neg\forall\lambda x.[f_w x]!\rightarrow[g_w x]!$; note that we cannot directly proceed further and define thus 'NotAll', which might be reason why the O-corner is nameless (cf. Béziau 2003); I will use

- we can add also:

$[[\mathrm{NotAll}\ f_w]\ g_w]$	$\Leftrightarrow_{df}$	$\neg\forall\lambda x.[f_w x]!\rightarrow[g_w x]!$	(btw. $\Leftrightarrow \neg[f_w \subseteq g_w]$)

II.4 The problem of existential import

- *existential import* of the term F in a statement P (i.e. F is a subconstruction of P) consists in P's entailing $\lambda w.\exists\lambda x[F_w x]$; if there is no F in W, P is false:

$$^0\lambda w.\neg\exists\lambda x[F_w x] \models {}^0\lambda w.\neg P_w$$

- if there is no F in W:
  - I and O are in natural sense false in W (but cf. below modal reading)
  - A and E are - on the *modern reading* - true in W, not false, because modern logic models A and E as *lacking existential import*
  - thus $A \not\models I$ and $E \not\models O$
- (later we put some light on this by inspecting truth-conditions of A, I, E, O)

II.5 Subalternation

- classical definition: A proposition Q is *subaltern* of P iff Q must be true if P is true, and P must be false if Q is false.
  (I.e. necessarily, $(P \rightarrow Q)$, and necessarily, $(\neg Q \rightarrow \neg P)$)

[Subaltern Q P]	$\Leftrightarrow_{df}$	$\forall \lambda w\, [P_w! \rightarrow Q_w!]$	(i.e. $P \models Q$)
[Superaltern P Q]	$\Leftrightarrow_{df}$	[Subaltern Q P]	

- since on the modern reading $A \not\models I$ and $E \not\models O$, *subalternation* does not generally hold
- the lack of subalternation invalidates contrariety and subcontrariety because the left conjuncts of their definiens assume $A \models I$ and $E \models O$, cf. below

II.6 Contrariety and subcontrariety

- classical definition: Two propositions are *contraries* iff they cannot both be true but can both be false. (I.e. necessarily, $\neg(P \wedge Q)$, and possibly, $(\neg P \wedge \neg Q)$)

- Sanford (1968, 96) noticed that the second condition cannot be omitted, as many authors do, because contradictions would be contrary as well

$$[\text{Contrary } P\ Q] \quad \Leftrightarrow_{\text{df}} \quad \forall\lambda w\,[P_w! \rightarrow \neg Q_w!] \wedge \exists\lambda w\,[\neg P_w! \wedge \neg Q_w!]$$

- classical example: *A* and *E*; no example on modern reading because $A \not\models \neg E$
- classical definition: Two propositions are *subcontraries* iff they cannot both be false but can both be true. (I.e. necessarily, $\neg(\neg Q \wedge \neg P)$, and possibly, $(P \wedge Q)$)

- again, the second condition cannot be omitted Sanford (1968, 96)

$$[\text{Subcontrary } P\ Q] \quad \Leftrightarrow_{\text{df}} \quad \forall\lambda w\,[\neg P_w! \rightarrow Q_w!] \wedge \exists\lambda w\,[P_w! \wedge Q_w!]$$

- classical example: *I* and *O*; no example on modern reading because $\neg I \not\models O$

II.7 Partiality and existential import

- $[F_w\ x]$ *v*-constructs nothing (thus e.g. $\lambda w.\forall\lambda x.[F_w\ x]\rightarrow[G_w\ x]$ is false – existential import), only if

    i. the property *F* is not defined for the given world *W*

    ii. the higher-order intension which should deliver property such as *F* is not defined for the given *W*

    iii. F_w *v*-constructs a partial class which is not defined for the value of *x*

- all three failures are fixed by employing !, $[F_w\ x]!$ *v*-constructs 0 on such *v*
- let us focus only on iii.: the *v*-improper construction $[F_w\ x]$ is a subconstruction of all constructions *A*-*O*, each of them is thus false on such *v*
- contradictoriness is preserved only if *O* and *E* start with ¬, e.g. $\lambda w.\ \neg\forall\lambda x.[F_w\ x]\rightarrow[G_w\ x]$ (for the case of *E*, this matches original Aristotle's claim in *De Interpretatione*)

III.

Modified modern reading of the Standard Square

III.1 The Standard Square of Opposition: two readings

- we are going to put forward the *modified modern reading* of the Standard Square (published already by W.H. Gottschalk 1953, 195, as the Square of Quaternality; cf. also Brown 1984, 315-316)
- it differs significantly from the *modern reading* in that also subalternation, contrariety and subcontrariety hold (= a better support for reasoning)
- the modified reading seems to be largely embraced in recent literature, yet without warning before a possible confusion between two dissimilar readings (a rare example of their distinguishing as Apuleian Square and the Logical Quatern can be found in Schang 2011, 294)
- on the modern reading, any categorical statement attributes something to the class C_i which is v-constructed, on a particular valuation v, by the body of the categorical i-statement (for i = A, I, E or O) (C_i is in fact a construction of a class)

III.2 The modern reading of the Standard Square and truth-conditions

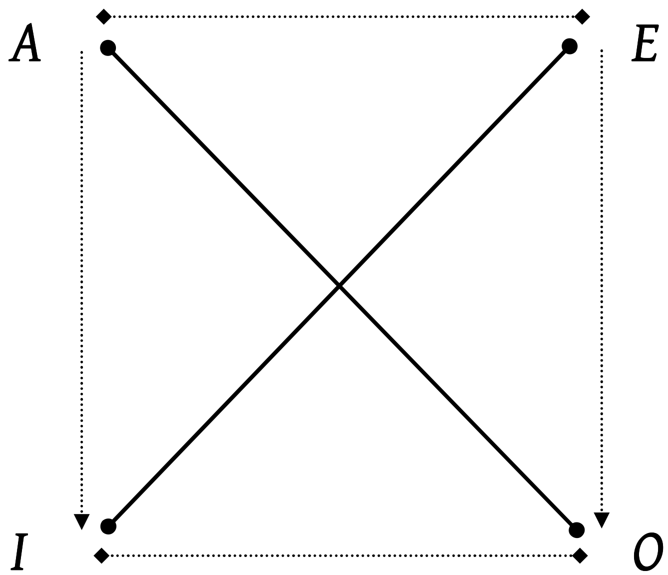
- let $Quant_i$ be any of the 4 classical quantifiers (∀, ¬∃, ∃, ¬∀); U is univ. of disc.

- $\boldsymbol{Quant_i C_i}$ is *true* (in *W*) if C_i (a class) is such and such (btw. $[\forall\ c] \Leftrightarrow_{df} [c{=}\mathrm{U}]$, etc.)

$\lambda w.\forall\lambda x.[F_w\ x]! \rightarrow [G_w\ x]!$ $\lambda w.\forall\lambda x.[F_w\ x]! \rightarrow \neg[G_w\ x]!$

$C_A{=}\mathrm{U}$ $C_E{=}\varnothing$

$\forall x\ (Fx \rightarrow Gx)$ $\forall x\ (Fx \rightarrow \neg Gx)$



$\exists x\ (Fx \wedge Gx)$ $\exists x\ (Fx \wedge \neg Gx)$

$C_I{\neq}\varnothing$ $C_O{\neq}\mathrm{U}$

$\lambda w.\exists\lambda x.[F_w\ x]! \wedge [G_w\ x]!$ $\lambda w.\exists\lambda x.[F_w\ x]! \wedge \neg[G_w\ x]!$

III.3 The source of invalidity of subalternation, contrariety, subcontrariety

- if there is no *F* in *W*: C_A=U, C_E=U, C_I=∅, C_O=∅
  (if there is an *F* in *W*, we usually get another quadruple of classes)
- such quadruple <U,U,∅,∅> preserves contradictories, but it does not preserve subalternation, etc., which are dependent on ***A*** |= ***I*** and ***E*** |= ***O*** (the entailment *A* |= *I* holds if C_A=U and C_I≠∅, i.e. C_I∈ Power(U)–{∅} in which U is included, this is not satisfied if there is no *F*; analogously for *E* |= *O*)
- clearly, the Standard Square is constructed in the modern reading only to preserve contradictories: regardless preserving subalternation etc., which would require strange existential assumptions (e.g. that only affirmative statements have existential import, Parsons 2014, 1.2; but recall that "Some women are not mothers" naturally entails existence of childless women, while "All chimeras are creatures" or "All ogres are ogres" do not naturally entail existence of chimeras or ogres)

III.4 The modified modern reading of the Standard Square

- recall that on the modern reading of the Standard Square, the 4 quantifiers ∀, ¬∃, ∃, ¬∀ (we have no single symbol for the even ones) apply to the heterogeneous collection of 4, not necessarily distinct, classes

C_A, C_I, C_E, C_O,

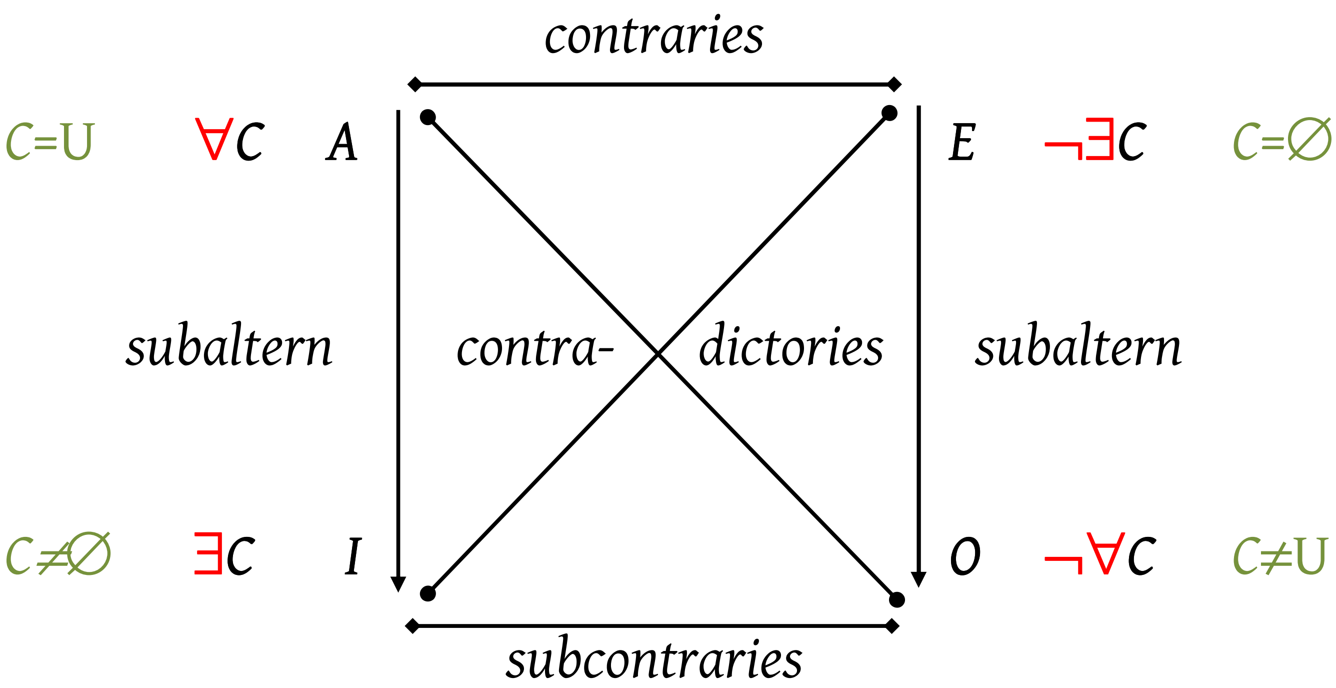
which are *v*-constructed by 4 distinct constructions

- on the *modified modern reading*, however, the vertices of the Standard Square are 'decorated' by a more tight class of constructions which have one and the same body, i.e. only 1 construction *v*-constructing 1 particular class

C

III.5 The modified modern reading of the Standard Square

- writing here simply $\forall C$, $\neg\exists C$, $\exists C$, $\neg\forall C$ because the particular form of the construction of *C* does not matter (but ***QuantC*** is still a categorical statement)



III.6 On the modified modern reading of the Standard Square

- an important claim: all classical rules, incl. subalternation, contrariety and subcontrariety, are confirmed for obvious reasons such as $\forall \subset \exists$ (thus $\forall C \models \exists C$)
- such reading is quite natural if we consider possible quantified forms of one statement such as, e.g., ‘*F* is *G*’:

“Every *F* is *G*”	$\lambda w.\forall\lambda x.[F_w\,x]!\rightarrow[G_w\,x]!$	normal categorical statement, CS
“Not some *F* is *G*”	$\lambda w.\neg\exists\lambda x.[F_w\,x]!\rightarrow[G_w\,x]!$	see the remark below
“Some *F* is *G*”	$\lambda w.\exists\lambda x.[F_w\,x]!\rightarrow[G_w\,x]!$	see the remark below
“Not every *F* is *G*”	$\lambda w.\neg\forall\lambda x.[F_w\,x]!\rightarrow[G_w\,x]!$	equivalent to normal CS

- note that the difference from the modern reading does not lie on surface linguistic form, but on the level of logical form represented in modern symbols

III.7 On the modified modern reading of the Standard Square (cont.)

- remark on *E*- and *I*-statements: “All chimeras are creatures” seems to entail “Some chimeras are creatures” regardless the existence of chimeras
- this is preserved with → instead of ∧ (thus these forms of *E*- and *I*-statements are justified)
- cf. also the example (using an empty subject term) by Paul of Venice: “Some man who is donkey is not a donkey” is true and follows from “No man who is donkey is a donkey”; the particular statement must be thus read as a conditional statement with → (compare it also with “Some non-identical objects are non-identical objects”)
- anyway, we should discriminate between 2 modern readings of the Square; note that putting only classical quantifiers in the vertices (as many authors do) means the *modified* modern reading

III.8 Gottschalk's Square of Quaternality

- Gottschalk (1953) proposed a Theory of Quaternality, which is a model of many possible squares of oppositions
- the basic form of the Square ("for quantifiers") resembles our modified reading (Gottschalk only renamed original relations: contradictory – "exactly one is true", contrariety – "at most one is true", subcontrariety – "at least one is true", subaltern – "if upper is true, then the lower is true")
- Gottschalk's Square of Quaternality for Restricted Quantifiers is supposed by him to be the Square in traditional form (*ibid.*, 195); however, it is obviously not:

for nonempty class s, $(\forall x \in s)(px)$ | $(\forall x \in s)\neg(px)$ | $(\exists x \in s)(px)$ | $(\exists x \in s)\neg(px)$

- realize that "$x \in s$" is a condition: if $x \in s$, then x is p or non-p; all formulas thus contain implicit $\rightarrow$ (not: some $\rightarrow$ and some $\wedge$)

III.9 Duality in the modified modern reading of the Standard Square

- Gottschalk (1953, 193) and lately e.g. Westerståhl (2005), D'Alfonso (2012) studied the Square using the notion of duality:

  A: original φ

  E: *contradual* of φ ('negation' of φ's variables)

  I: *negational* of φ (exchange of φ's dual constants and 'negation' of φ's variables)

  O: *dual* of φ (exchange of φ's dual constants, e.g. $\forall$ for $\exists$, $\vee$ for $\wedge$, $\rightarrow$ for $\neg\leftarrow$)

- but Gottschalk's duality and contraduality work only for the modified reading, not for the modern reading of the Square; for instance, Gottschalk's contradual of $\forall x(Fx \rightarrow Gx)$ is $\forall x(\neg Fx \rightarrow \neg Gx)$, not the familiar $\forall x(Fx \rightarrow \neg Gx)$; the dual of $\forall x(Fx \rightarrow Gx)$ is $\exists x \neg(Fx \leftarrow Gx)$, i.e. $\exists x\, (\neg Fx \wedge Gx)$, not the familiar $\exists x\, (Fx \wedge \neg Gx)$), etc.

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III.10 Duality in the modified modern reading of the Standard Square

- Brown (1984), Westerståhl (2005), D'Alfonso (2012) in fact suggested a solution to this problem, they introduced "inner negation" ("post-complement") which places properly inside the formula (we get another notion of dual)
- then, the dual of *Quant*(*F*,*G*) is its outer and inner negation, i.e. ¬*Quant*(*F*,¬*G*) (it seems that the inner negation is not a negation but the function Non-, yet Non- is interdefinable with the "verb-phrase"-negation)
- D'Alfonso (2012) defines (cf. Brown 1984, 309):

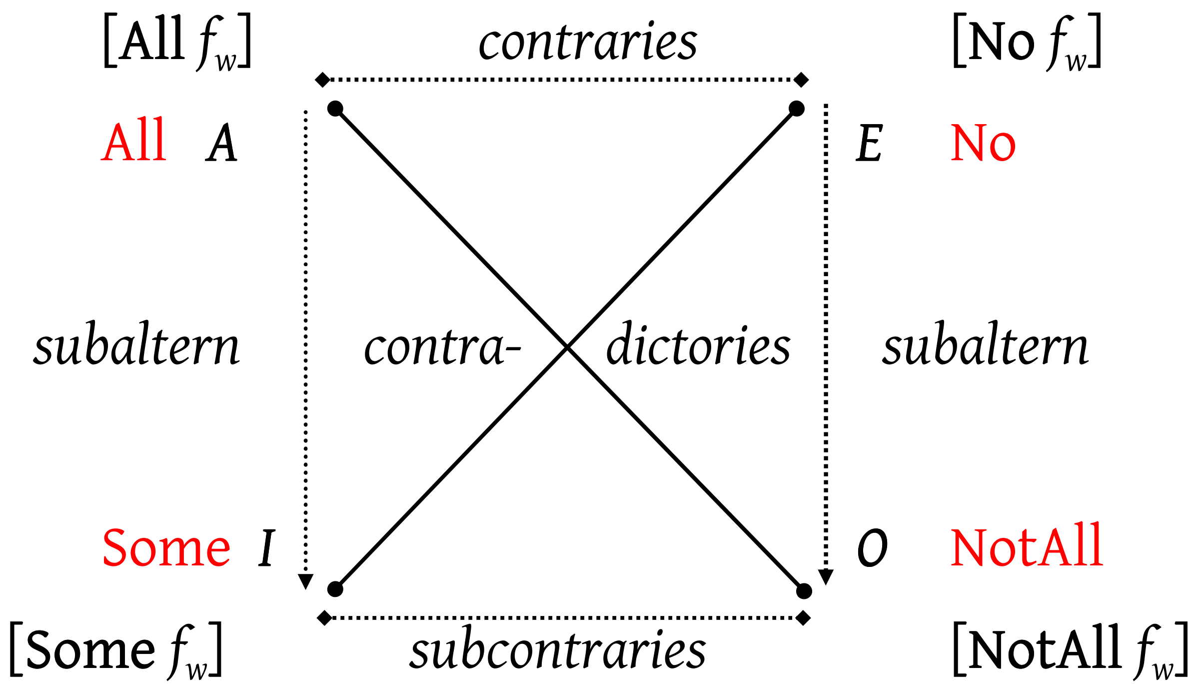
  outer negation: ¬*Quant*(*F*,*G*) $=_{df}$ {Power(U)^2–*Quant*(*F*,*G*)} (it should be rather {Power(U)^2–*Quant*}(*F*,*G*))

  inner negation: *Quant*¬(*F*,*G*) $=_{df}$ {*Quant*(*F*,U–*G*)} (it should be rather {Quant(*F*,U–*G*)})

  dual: (*Quant*(*F*,*G*))$^{\text{dual}}$ $=_{df}$ ¬*Quant*¬(*F*,*G*)

III.11 Another modified modern reading of the Standard Square?

- it is rather confusing to put generalized quantifiers in the vertices because there is no important logical similarity to the modified modern reading (cf. below)



- recall that [All f_w] $\Leftrightarrow_{df}$ $\lambda g.\forall\lambda x.[f_w x]! \rightarrow [g_w x]!$, whereas [All f_w]/$(o(o\xi))$

III.12 Another modified modern reading of the Standard Square? (cont.)

- though [All f_w], [No f_w] etc. seems to be applicable to one and the same G_w (a similarity with the modified reading), it follows from definitions of All, No etc. that subalternation, contrariety, and subcontrariety cannot hold
- to verify it, realize that the function All maps the class F_w to the class of classes in which F_w is included; the function Some maps class F_w to the class of classes which overlaps with F_w (both characterizations are in Tichý 1976 and can be adapted for No and NotAll)

for instance, let U={α,β} and F_w={α}; then [All F_w] *v*-constructs {{α}, U} and [Some F_w] *v*-constructs {{α}, U}; thus, [All F_w] ⊆ [Some F_w];

however, if there is no *F* in *W* (the value of *w*), [All F_w] *v*-constructs {∅,{α},{β},U}, but [Some F_w] *v*-constructs ∅ (of the appropriate type), thus [All F_w] ⊄ [Some F_w], i.e. subalternation is lost (analogously for No and NotAll)

IV.

Modified reading of the Modal Square of Opposition

IV.1 The modified and normal reading of the Modal Square

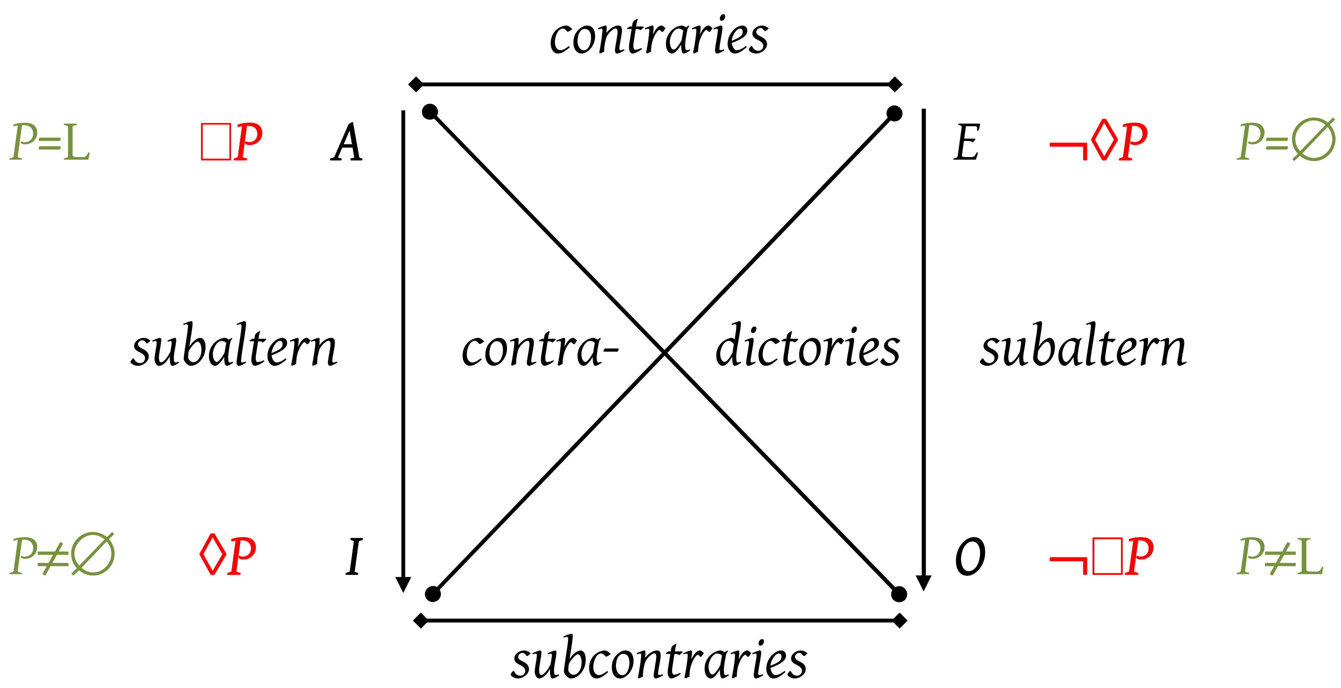
- we are going to study two readings of the Modal Square of Opposition, i.e. the Square whose vertices are 'decorated' by ('alethic') modal statements

  the reading of Square as concerning modal (and even deontic) notion appears in Leibniz (cf. Joerden 2012), but it is known already in the 13th century (cf. Knuuttila 2013, Ueckelman 2008)

- each *modal operator* M_i (i.e. , ¬◊, ◊, ¬) is a 'quantifier' for propositions, it is of type (ο(οω)) (a class of classes of worlds, i.e. a 'predicate' applicable to classes of worlds)
- we start with the *modified reading* which deploys statements of form M_iP, whereas (the construction of) P is *one and the same* (Gottchalk (1953, 195), Blanché (1966), see also Dufatanye (2012)
- this reading is natural when one considers various modal (*de dicto*) qualifications of one given statement ("Necessarily, all ravens are black", "Possibly, ale ravens are black", ...); necesses est esse/impossible est esse/possible est ese/possible non est esse; !that some *F* is *G* is necessary" – de dicto reading

IV.2 The modified reading of the Modal Square

- let P be any proposition (i.e. a class of Ws) constructed by a categorical statement
- let L be the universal class of possible worlds (in this context, ∅ is the empty class of worlds)
- $M_i P$ is *true* (*in W*) if P is such and such (btw. $\Box p \Leftrightarrow_{df} [p=\mathrm{L}]$, etc.)



IV.3 The modified reading of the Modal Square and deduction

- on this modified reading, the Modal Square is obviously nothing but a *type-theoretic variant of the Standard Square*, only the type of quantifiers and classes is altered (anticipated by Gottschalk 1953, 195)
- thus, not only contradictories, contrapositions and obversions, but also subalternation, contrariety and subcontrariety hold (to illustrate, $\Diamond$ is Power(L)–{∅}, thus L is its member; consequently, $A \models I$ and we get subalternation)

V.

Modal reading of categorical statements

V.1 Modal reading of categorical sentences

- this *novel* modern reading of the Modal Square is based on the assumption that in common language we often understand categorical sentences:

  "(Every) *F* is *G*"

  as expressing a certain necessary connection between *F* and *G*
- this is sometimes made explicit by inserting "*by definition*" or even "*necessarily*", cf. e.g. "A horse is, by definition, an animal"
- independently on mediaeval debate, Tichý (1976, §42) suggested reading sentences with such 'implicit modalities' as talking about so-called *requisites*; I will adopt and extend his proposal

V.2 Requisites: the philosophical picture

- Tichý (1976, §42) defined the notion of requisite and essence for both individual 'concepts' (individual offices) and for properties (the two notions differ)
- he preserved the idea that *essence* is *everything* that is *necessary* to become such and such; essence is a certain collection of requisites
- a *requisite* is *one of necessary conditions* (properties) for an object to be such and such; an object must possess that *property* to become such and such
- we may say that a requisite is an 'intrinsic property' (not in Lewis' sense) of a thing, while it is an 'extrinsic property' of an object possibly being that thing
- example: *(BE) WINGED* is a requisite of the individual 'office' *PEGASUS*, while the very same property is a(n external) property for any particular individual

V.3 Requisites of properties: definition

- in the case of properties, the requisites are particular ‘subproperties’ of a property
- example: *(BE AN) ANIMAL* is one of many requisites of *(BE A) HORSE*

- **Requisite**/$(o((o\xi)\omega)((o\xi)\omega))$ (a total relation between ξ-properties)

$$[\mathbf{Requisite}\ g\ f] \quad \Leftrightarrow_{df} \quad \forall\lambda w.\forall\lambda x.[f_w\ x]! \rightarrow [g_w\ x]!$$

- entailment between propositions P and Q, based on the fact $P \subseteq Q$ (i.e. $P|=^{\pi}Q$) is a medadic case of *entailment between properties*; realize thus that, in any world W, the extension of *F in W* $\subseteq$ the extension of *G in W*, i.e.:

$$[\mathbf{Entails}\ f\ g] \quad \Leftrightarrow_{df} \quad [\mathbf{Requisite}\ g\ f]$$

V.4 Potentialities

- to complete the investigation of the Square, we need another notion which would be comparable with the notion of requisite; I call it "*potentiality*" (Aristotle?)
- to explain: an individual which can possess the property *(BE A) HORSE* has to be an animal, i.e. it instantiates the property *(BE AN) ANIMAL*; but the property *(BE A) HORSE* admits the individual being white or fast, etc.; the property *(BE) WHITE* is thus mere potentiality

- **Potentiality**/$(o((o\xi)\omega)((o\xi)\omega))$ (a total relation between ξ-properties)

$$[\mathbf{Potentiality}\ g\ f] \quad \Leftrightarrow_{\mathrm{df}} \quad \exists\lambda w.\exists\lambda x.[f_w\ x]\wedge[g_w\ x]$$

- again, the definiendum is, if closed by λw, a modal version of a categorical statement

V.5 On the relationship of requisites and potentialities

- usually, $F_w \subseteq G_w$ for any W, i.e. G is a requisite of F
- but: let $[[f \oplus g]_w\, x] \Leftrightarrow_{df} [f_w\, x]! \wedge [g_w\, x]!$ (the 'conjunction' of properties)

  generally, ($G \oplus$Non-G) need not to be a requisite of F because there can be Ws such that $(G \oplus \text{Non-}G)_W \subset F_W$

- a potentiality G of a property F is ($=_{df}$) an *accidental property for* (JR 2007) every bearer of F
- a requisite G of a property F is ($=_{df}$) an *essential property for* (JR 2007) every bearer of F

- of course:

  $\neg$[Requisite $g\, f$] $\Leftrightarrow \neg\forall\lambda w.\forall\lambda x.[f_w\, x]! \rightarrow [g_w\, x]!$ $\Leftrightarrow \exists\lambda w.\exists\lambda x.[f_w\, x] \wedge \neg[g_w\, x]$

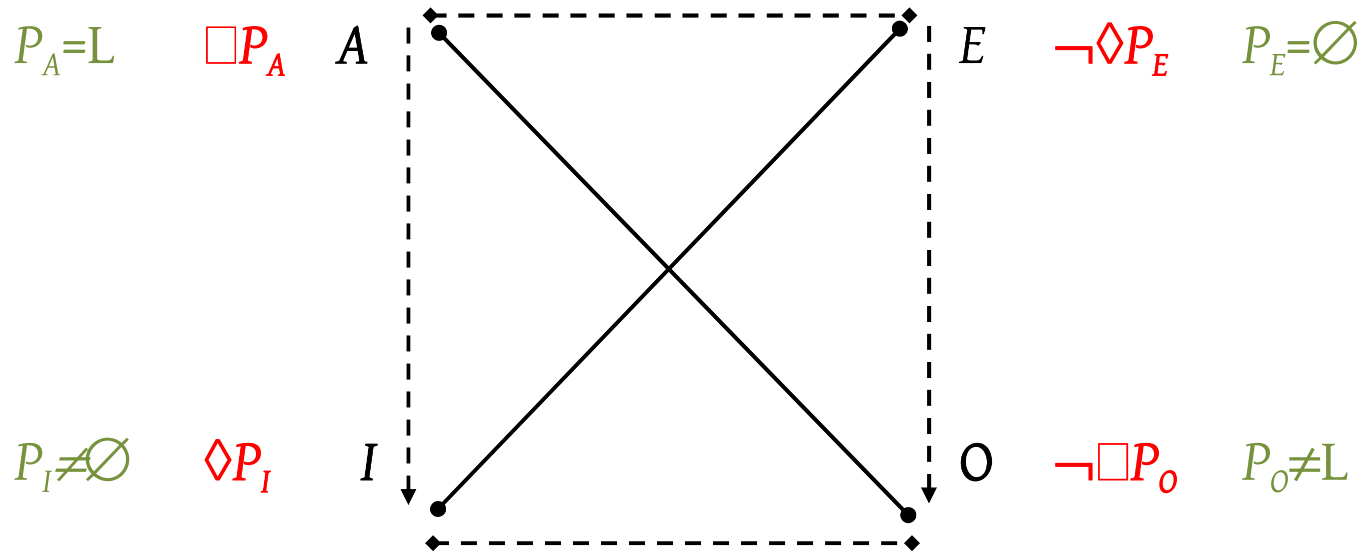
  $\neg$[Potentiality $g\, f$] $\Leftrightarrow \neg\exists\lambda w.\exists\lambda x.[f_w\, x] \wedge [g_w\, x]$ $\Leftrightarrow \forall\lambda w.\forall\lambda x.[f_w\, x]! \rightarrow \neg[g_w\, x]!$

VI.

Modern reading of the Modal Square of Opposition

VI.1 Modern reading of the Modal Square

- ‘decorating’ vertices of the Square by *modal versions of categorical statements*
- let P_i (i.e. a class of *W*s) be any proposition constructed by a categorical statement
- $\boldsymbol{M_i P_i}$ is *true* (in *W*) if P_i is such and such (P$_i$ is in fact a construction of a proposition)
- (the picture is here only to notice the difference from ‘modified reading’ with just one *P*)



VI.2 Normal reading of the Modal Square with requisites / potentialities

$\lambda w'.\forall.\lambda w''.\lambda w.\forall\lambda x.[F_w x]!\rightarrow[G_w x]!$

"(Necessarily,) every F is G."

λw [Requisite G F]

$\Box\forall x\,(Fx\rightarrow Gx)$

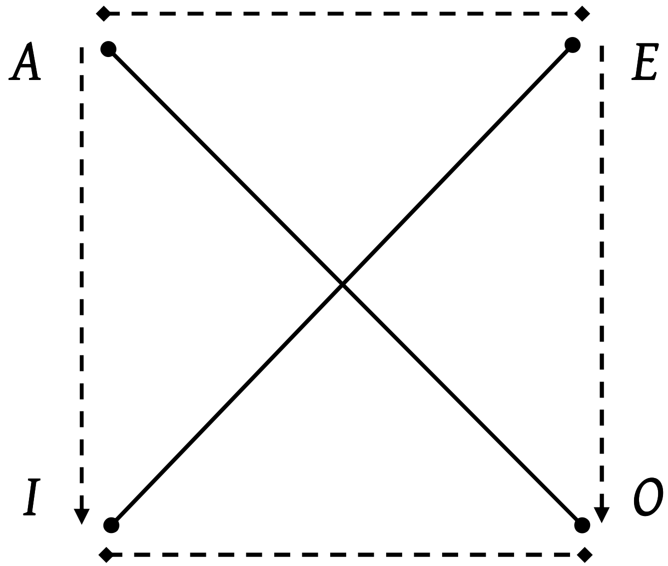
$\lambda w'.\forall.\lambda w''.\lambda w.\forall\lambda x.[F_w x]!\rightarrow\neg[G_w x]!$

"(Necessarily,) none F is G."

$\lambda w.\neg$[Potentiality G F]

$\neg\Diamond\exists x\,(Fx\wedge Gx)$

$\Box\forall x\,(Fx\rightarrow\neg Gx)$



$\neg\Box\forall x\,(Fx\rightarrow Gx)$

$\Diamond\exists x\,(Fx\wedge\neg Gx)$

$\Diamond\exists x\,(Fx\wedge Gx)$

λw [Potentiality G F]

"(Possibly,) some F is G."

$\lambda w'.\exists.\lambda w''.\lambda w.\exists\lambda x.[F_w x]!\wedge[G_w x]!$

$\lambda w.\neg$[Requisite G F]

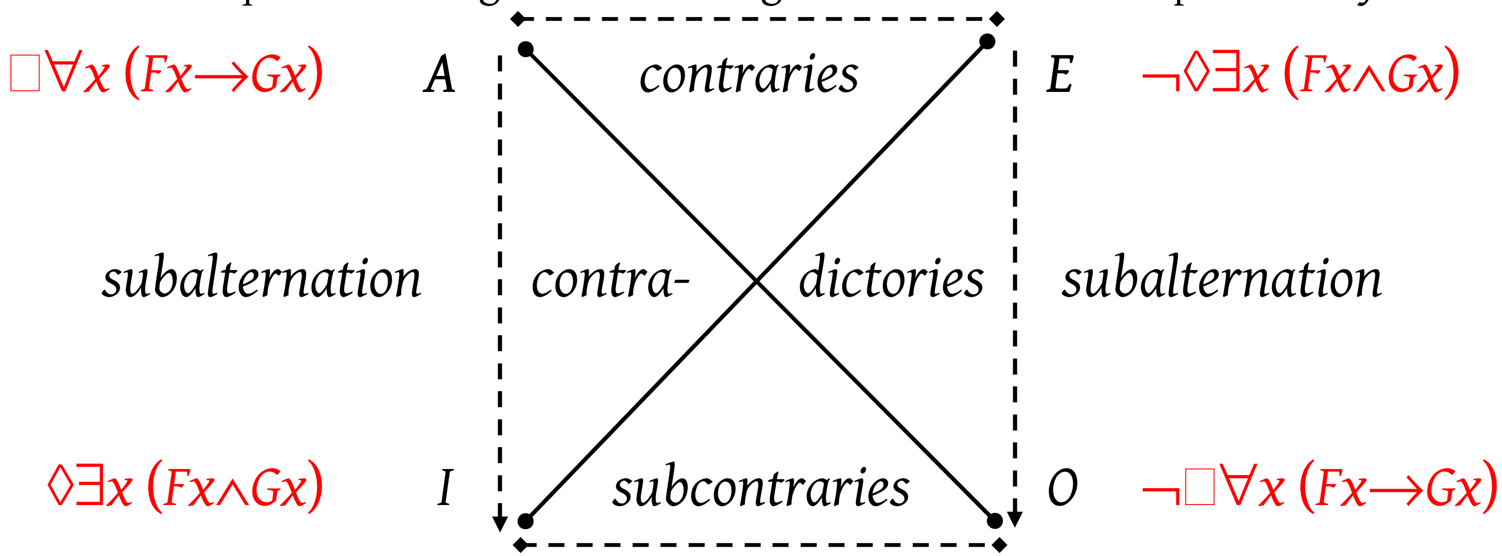
"(Possibly,) some F is not G."

$\lambda w'.\exists.\lambda w''.\lambda w.\exists\lambda x.[F_w x]!\wedge\neg[G_w x]!$

VI.3 The Modal Square: truths and falsities

- consider a) that the $\underline{A}$-statement is 1 (i.e. true); then, its modification (not written below) I is also 1, but its modifications E and O are both 0 (i.e. false)
- analogously, b) $\underline{E}_W=O_W=1$, but $A_W=I_W=0$; c) $\underline{I}_W=O_W=1$, but $A_W=E_W=0$; d) $\underline{O}_W=I_W=1$, but $A_W=E_W=0$
- (examples of 4 intuitively valid sentences; for each, please construct its modification in the 3 remaining vertices)

"Being an animal is a requisite of being a horse" "Being non-identical is not a potentiality of being a horse"



"Being black is a potentiality of being a horse" "Being fast is not a requisite of being a horse"

VI.4 The Modal Square: the lack of actual existential import

- using this modal reading of categorical sentences, we can immediately resolve the puzzle concerning existential import and *A*- and *O*- statements:

“Every griffin is a creature” is *true* (in every *W*), *despite* that there are no griffins; the sentence has no existential import

“Some griffin is not a creature” is *false* (in every *W*), *regardless* that there are no griffins; the sentence has no existential import

- clearly, modal categorical statement have *no actual existential import*
- because of the lack of existential import, *weakened modes of syllogisms* (cf. e.g. Darapti: “All *H* are *F*”, “All *F* are *G*”, “Therefore, some *F* are *G*”) are (with few exceptions deploying ‘contradictory beings’) *valid* on this modal reading

VI.5 The Modal Square: subalternation and existential import

- generally, $A \not\models I$, $E \not\models O$; consequently, *subalternation does not hold* (similarly as in the Standard Square)
- the reason of the invalidity of $A \models I$ and $E \models O$ consists in that there are *properties that can have no instance in any possible world* (i.e. across the whole logical space)

- although the *A*-statement “Necessarily, everybody who shaves all and only those who do not shave themselves is a barber” is true, the corresponding *I*-statement “Possibly, somebody who shaves all and only those who do not shave themselves is a barber” is not (cf. also contrariety below)

VI.6 The Modal Square: subalternation and existential import (cont.)

- analogously, the *E*-statement "Necessarily, no non-identical individuals are identical individuals" (¬*G*=*F*) is true, while the corresponding *O*-statement ("Possibly, there are non-identical individuals who are not identical individuals") not (if not using the 'if-reading' of the sentence)

- there is a hypothesis that mediaeval logicians purposely ignored these '*contradictory beings*' (in fact so-called empty properties, JR 2007) which have no possible instances
- (this ignorance would be much more tolerable attitude in the case of modal categorical statements we just discuss than in the case of ordinary categorical statements)

VI.7 The Modal Square: contraries and subcontraries

- similarly as for contraries and subcontraries in the Standard Square above
- an instance of classical definition of contrariety:

$$\lambda w'.\forall\lambda w\,[A_w! \rightarrow \neg E_w!] \wedge \exists\lambda w\,[\neg A_w! \wedge \neg E_w!]$$

- if A_W=1, then $\neg\exists\lambda w.\neg A_w$; the right conjunct of the (instance of the) definiens is thus not satisfied, contrariety does not generally hold

- an instance of classical definition of subcontrariety:

$$\lambda w'.\forall\lambda w\,[\neg I_w! \rightarrow O_w!] \wedge \exists\lambda w\,[I_w! \wedge O_w!]$$

- if I_W=0, then $\lambda w'.\neg\exists\lambda w I_w$; the right conjunct of the (instance of the) definiens is thus not satisfied, subcontrariety does not generally hold

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VI.8 The Modal Square: analytic statements and contraries/subcontraries

- Sanford (1968, 95) noticed that contraries cannot be both false if *A* is a necessary categorical statement ("All squares are rectangles"); analogously for *I* and *O*, since truth of *O* amounts to falsity of *E*
- but we may note that it is evident that this feature is peculiar to statements which are logically equivalent to their modal versions: "All squares are rectangles" ⇔ "Necessarily, all squares are rectangles", "No squares are round" ⇔ "Necessarily, no squares are round"

VI.9 Existential import and affirmative/negative statements

- Parsons (2008, 2012) maintains that only affirmative statements have an existential import
- one can reject such view (e.g. Westerståhl 2005) as inconvenient from the viewpoint of modern logic
- we can reject it also for the reason that justification of this view is apt only for modal version of the Square
- the justification mentioned by Parsons: assume I_W=0, its contradictory E_W=1; $E \models O$, thus O_W=1; then, O's contradictory A is such A_W=0; hence, non-existence of F makes A and I false, while E and O not
- note that this reasoning is based on the validity of subalternation, cf. $E \models O$, and subalternation $A \models I$ implies existential import of A (circulus vitiosus)

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VI.10 Existential import and affirmative/negative statements (cont.)

- on the modern reading of the Modal Square, the only reason of invalidity of *I*-statement is that it treats 'contradictory being' - there is no individual which can bear the potentiality *G* in question

  (another reason: interdependencies of properties – (*BE*) *WOMAN* is not potentiality of (*BE*) *BACHELOR*)
- if I_W=0, then *G* is not a potentiality, i.e. E_W=1, *G* is not even a requisite, thus O_W=1 and A_W=0
- recall that modal (*de dicto*) categorical statements do not have existential import; and the just given reasoning does not push us to claim that it has

VII.

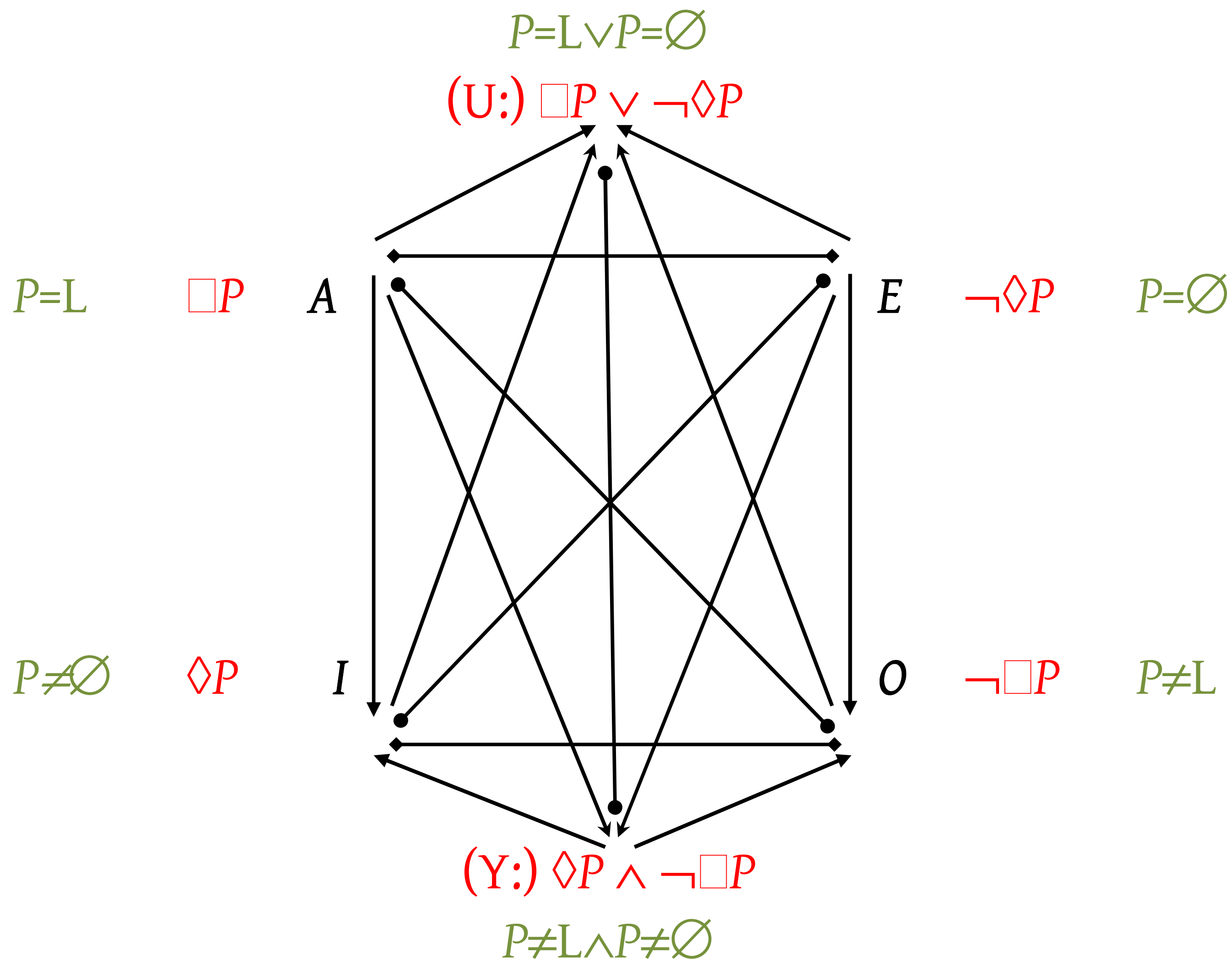
Modal Hexagon of Oppositions

VII.1 The Modal Hexagon of Oppositions: two new quantifiers

- to remind the reader, W.H. Gottschalk (1953, 195) introduced and investigated (Modal – 'Alethic') Hexagon of Oppositions with 2 new quantifiers; the theory was independently discovered and developed by Robert Blanché (1966) who defined the Y operator already in (1952, 370)
- U, i.e. $\Box P \lor \neg\Diamond P$, is a genuine new quantifier, it is the class {L,∅}
- Y, i.e. $\Diamond P \land \neg\Box P$, is a new quantifier, it is the class Power(L)–{L,∅}

- each of them 'governs' its half of the diagram (the analytic and contingent ones)
- U is well known in philopas *analytic* (then: $\Box$ / $\neg\Diamond$ is analytically true / false) or *non-contingency* (Gottschalk 1953, Blanché 1966, Béziau 2012, Dufatanye 2012) or determined (Joerden 2012)
- Y is can aptly be called (purely) *contingent* (Blanché 1952, 1966, Gottschalk 1953; see Moretti 2012)
- $\Box P$ = *necessary*; $\neg\Diamond P$ = impossible; $\Diamond P$ = *possible*; $\neg\Box P$ = nonnecessary

VII.2 Modal Hexagon of Oppositions (with one *P*)



VIII.

Conclusions

VIII. Conclusions

- there are 2 modern readings of the *Standard Square of Opposition*:
  i. the well-known one (the Square of modern logic textbooks), for which subalternation, contrariety, subcontrariety do not hold
  ii. the less known one (the Square of Quaternality), for which all classical relations, even subalternation, contrariety, subcontrariety hold
- a shift from i. and ii. can explain some confusions occurring in the literature
- there are 2 modern readings of the *Modal Square of Opposition* (with modal versions of categorical statements):
  iii. the one which is nothing but a general form of the Square ii.; subalternation, contrariety and subcontrariety hold
  iv. the one which is the modal version of the well-known non-modal Square i.

VIII. Conclusions (cont.)

- in the case of the Square iv., subalternation, contrariety, subcontrariety do not hold only because of existence of rare *properties which cannot be instantiated*; admittedly, these properties may be dismissed by someone as non-properties or *'contradictory beings'*, thus subalternation, contrariety and subcontrariety would generally hold
- the Square iv. is interesting as an interpretation of the Square also because pre-modern tendencies to adopt some form of *essentialism*; the shift from i. to iv. can thus nicely explain oppositions if we shift from normal to 'mythological' discourse (cf. Nelson 1954, 409)
- the Square iv. is interesting also for is its lack of *actual existential import*, which validates also *weakened forms of syllogisms* (as mediaeval logicians held)

IX. Some prospects

- recall that the positive of the above proposal is its logic which has
  i. a transparent treatment of modality (explicit quantification over possible worlds etc.),
  ii. a clear philosophical background (only a small fragment of it was presented above – the doctrine of requisites)
  iii. is extremely flexible, since it is based on (typed) λ-calculus
- we can then fruitfully compare the above results (which consists mainly in disambiguation of the discussion about the basic Square) with the claims of mediaeval logicians as well as Aristotle and also some contemporary writers
- for instance, one may investigate modalities *de re* in the Square i.

IX. Some prospects (cont.)

- we can examine other parts of the ancient and mediaeval logic, e.g. *modal syllogistics*
- to illustrate, let us resolve the notorious puzzle of LXL Barbara/XLL Barbara, whereas the former is valid according to Aristotle, but the letter is not
- the two Barbaras use modal versions of categorical statements, where L means necessity and X means assertoric modality (=no operator)

LXL Barbara	XLL Barbara
"Necessarily, *G* belongs to all *F*"	"*G* belongs to all *F*"
"*F* belongs to all *H*"	"Necessarily, *F* belongs to all *H*"
"Therefore, necessarily *G* belongs to all *H*"	"Therefore, necessarily *G* belongs to all *H*"

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IX. Some prospects (cont., cont.)

- using the notion of requisite, but utilizing here its equivalent, we get:

LXL Barbara:

$$\lambda w' \forall \lambda w \ [G_w \subseteq F_w]$$
$$\lambda w' \ [H_w \subseteq G_w]$$
$$\therefore \lambda w' \forall \lambda w \ [H_w \subseteq F_w]$$

XLL Barbara:

$$\lambda w' \ [G_w \subseteq F_w]$$
$$\lambda w' \forall \lambda w \ [H_w \subseteq G_w]$$
$$\therefore \lambda w' \forall \lambda w \ [H_w \subseteq F_w]$$

- LXL Barbara is valid; consider e.g. *H*=(*BE*) *HUNGRY*, *G*=(*BE*) *HUMAN*, *F*=(*BE*) *PRIMATE*; the conclusion is obviously false because (e.g.) hungry dogs can be counted among primates – but because of existence of *W*s in which exclusively some dog is hungry the second premise cannot be true
- LXL Barbara is invalid; consider e.g. *H*=(*BE*) *HUMAN*, *G*=(*BE*) *PRIMATE*, *F*=(*BE*) *HUNGRY*; if primates are hungry, the first premise is true, while the conclusion is not

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