Fitch's Knowability Paradox

and Typing Knowledge

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Logika: systémový rámec rozvoje oboru v ČR a koncepce logických propedeutik
pro mezioborová studia (reg. č. CZ.1.07/2.2.00/28.0216, OPVK)

doc. PhDr. Jiří Raclavský, Ph.D. (*raclavsky@phil.muni.cz*)

Department of Philosophy, Masaryk University, Brno

Abstract

It is already known that Fitch's knowability paradox can be solved by typing knowledge. I differentiate two kinds of such typings, Tarskian and Russellian, and focus on the latter which is framed within the ramified theory of types. My main aim is to offer a defence of the approach against a recently raised criticism. The key justification is provided by the Vicious Circle Principle which governs the very formation of propositions and thus also intensional operators, including the operator of knowledge.

Content

I. Introduction to the typing approach to Fitch's knowability paradox; its basic criticism; the refutation of the criticism

II. Crucial ideas of Russellian typing knowledge; the typing rule; blocking the reductio of Fitch's knowability paradox; an invalid rule

III. Original revenges by Carrara and Fassio, Williamson and Hart; 2 'monadic' readings of the (Hart's) revenge, 4 'dyadic' readings of the revenge; an invalid rule

IV. Brief conclusion

3. Typing knowledge need not to protect verificationism (Objection 1)

- the epistemic optimism known as *verificationism*:

(**Ver**) $\forall p\ (p \supset \Diamond \mathrm{K}p)$ "every truth is knowable"

- some critics show that some *other* paradox than FP is not blocked by typing and complain that the approach thus *does not protect verificationism* (*cf*., e.g., Jago 2010, Florio and Murzi 2009)
- my reply: typing knowledge and protecting verificationism are independent enterprises
- I will not study the paradoxes distinct from FP

Logika: systémový rámec rozvoje oboru v ČR a koncepce logických propedeutik pro mezioborová studia (reg. č. CZ.1.07/2.2.00/28.0216, OPVK)

4. Typing of the knowledge operator K is *ad hoc* (Objection 2)

- Carrara and Fassio (2011) published an extensive criticism of the approach
- their main idea is that *typing of K is ad hoc*, i.e. it has no other, independent reason than to solve the paradox
- other theoreticians (*cf.*, e.g., Paseau 2008) seem to suggest a similar criticism (note: Paseau is in fact neutral, he only discusses a possible criticism)

- my leading idea: to the large extent, the criticism is in fact *misguided* because its target is something other than a 'full-blooded' typing within RTT
- one must *distinguish here Russellian and Tarskian typing* (it was perhaps Church 1973-74 who seem to confuse, unintentionally, the two)

5. Tarskian typing (1/2)

- the method of Tarskian typing (called simply 'typing') is known from recent theories of truth (see, e.g., Halbach 2011)
- the obvious inspiration is Tarski (1933/1956), his hierarchy of languages and hierarchy of T-predicates
- (for some problems with combining typing of T, $\Box$ and K see Halbach 2008, Paseau 2009)
- the formulas (not propositions) such as '$\mathrm{K}_1\ulcorner p_0\urcorner$' or sometimes '$\mathrm{K}_1 p_0$' involve the predicate 'K_n' applicable to (the names of) sentences/formulas
- the subscript '$_n$' in 'K_n' indicates the order (alternatively: type, level) of the predicate and, mainly, the resulting order of the whole sentence/formula

6. Tarskian typing (2/2)

- this kind of typing has officially no other reason than to solve the paradox (it is thus *ad hoc*)
- my remark: pace Carrara and Fassio, this is *not an entirely idle reason* especially when one wants to provide a formally correct explication of a notion (T or K)
- another justification of Tarskian typing seems to be problematic: the stratification of T-predicates corresponds to the hierarchy of languages, whereas the meta-languages are tools for speaking about the object-languages; it is difficult to find an analogy of this for the case of K-operators

7. Russellian typing (history)

- Russellian typing was firstly exposed by Russell (*cf.* 1903, 1908, 1910-13), though he never typed belief or knowledge; this was suggested by Church
- Church first (1945) mentioned a solution of FP by Tarski's or Russell's method; but he himself solves the Paradox of Bouleus by Tarskian typing
- *Church's ramified theory of types* (1976) (i.e. not his simple TT, not his simple Russellian TT, but his theory of *r*-types) was firstly applied to FP by Linsky (2009), *cf.* also Giaretta (2009)
- a bit unfortunately, Linsky (2009) paid only little attention to the justification of the method

- remember: the only RTT adopted in this talk is Churchian RTT

8. Russellian typing (intensional entities, types, orders)

- *intensional entities* have not extensional, but intensional identity criteria
 (two such entities can be equivalent/congruent but not identical)
- e.g. *propositions* (i.e. structured meaning of sentences, not mere concatenations of letters!) and *intensional operators* operating on propositions (e.g. knowledge, belief, ...)

- the key feature of RTTs: every intensional operator such as K has a number of type (order) variants, e.g. K^1, K^2, ..., K^n
  (the typing rule will be exposed later on)

9. Russellian typing (types, orders, cumulativity)

- *type* can be described as a collection (i.e. set) of objects of the same nature
- *extensional types*: e.g. the type of individuals, of truth-values, of truth-functions , ...
- *intensional types*: e.g. the type of propositions, the type of monadic propositional operators, ...
- intensional types are *ramified*, i.e. divided into *order* variants, having thus (e.g.) the type of 1st-order propositions, the type of 2nd-order propositions, ..., the type of *n*-order propositions ($1 \leq k \leq n$)
- in Churchian RTT, we have *cumulativity* (Church 1976):

  Every entity of order *k* (i.e. belonging to the *k*-order type in question) is also of order k+1.

10. The Principle of Specification

- the formation of any entity is to be noncircular (*cf.* Whitehead, Russell 1910); put in the form of the *Principle of Specification*:

No entity can be fully specified in terms of itself.

- to fully specify a function, one must firstly determine all its arguments and values; this would be impossible if the function itself were among the arguments or values

- the Principle of Specification entails various *Vicious Circle Principles*, ***VCPs*** (e.g., the Extensional VCP implemented in Church's simple theory of types)

11. Formation of intensional entities, Intensional VCP

- intensional entities (propositions, intensional operators) are structured; they may contain variables for intensional entities, they can be substituted for by some entities (and *vice versa*)
- each variable is defined in terms of the type of objects, whereas the type is the range of the variable
- the formulation of the *Intensional VCP* adapted from (Russell 1908; Russell often used informal, non-technical presentation of VCP):

Any intensional entity containing a variable cannot be in the range of the variable, it is thus of (i.e. belongs to) a higher type. (VCP)

- e.g. the composed proposition (*...p...*) cannot be in the range of *p*

12. Ramification

- Russell discovered the validity of VCP for a bit different reason; his Paradox of Propositions (1903) showed him that, for any totality of propositions, there are propositions talking about (quantifying over) the totality, while it would be paradoxical if the propositions were members of the totality; thus propositions cannot form one all-inclusive 'totality' of propositions; Russell then often spoke about 'completed totality' of propositions of a particular order, which is presupposed by some higher-order propositions

- note thus carefully: *the typing of intensional entities is justified by the rules (esp. VCP) for their non-circular formation (individuation)*

13. Four kinds of justification

- being inspired by thoughts by Williamson (2000), Carrara and Fassio (2011), let us distinguish typing justified by (differences in) *content* and typing justified by (differences in) *states of knowledge* (Paseau: *epistemic access*)
- Paseau (2008) distinguishes also *logical* and *philosophical* versions of the two kinds of justification, thus we have a quadruple of justifications
- it is readily seen that Tarskian typing is justified only by distinct content (viz. presence/absence of K) for logical reasons (viz. avoiding a paradox)
- as showed above, *Russellian typing is well justified both by logical and philosophical reasons as regards the distinct content*, *cf.* the conditions on successful formation of propositions, VCP, and avoiding paradoxes (*cf.* also the next example)

Logika: systémový rámec rozvoje oboru v ČR a koncepce logických propedeutik pro mezioborová studia (reg. č. CZ.1.07/2.2.00/28.0216, OPVK)

14. A confusion about epistemic content (Objection 3)

- let us call propositions having K as its (main) constituent *epistemic propositions* (e.g. "Xenia believes that Fido is a dog") and propositions having no such constituent *basic propositions* (e.g. "Fido is a dog")
- Carrara and Fassio 2011 object to (Russellian?) typing knowledge that the borderline between epistemic and non-epistemic propositions is fuzzy - the proposition "Xenia is lying in bed" is also epistemic, they say, it informs us that Xenia does not know what happens in the kitchen
- the critics misunderstood the fact that one types propositions in accordance to the presence or absence of K-operator in a proposition; it has nothing to do with the logically independent information a hearer has

15. About the next slides

- I am going to show a philosophical and logical reason for Russellian typing motivated by differences of epistemic accesses (states of knowledge)
- before, I formulate the typing rule for proposition in order to show how Russellian typing blocks FP
- we will then discuss an objection that the typing does not work for some technical reason
- but, mainly, we meet a problem for the Russellian typing, whereas its the solution of the problem provides a justification related to epistemic matters

16. Russellian typing (typing rule for propositions)

- the *typing rule for propositions* (we speak only about orders because we already know that we discuss the type of propositions):

The lowest order of any proposition involving no intensional operator is 1.

Let p^k be any proposition of order k, for $k \geq 1$.
The lowest order of an intensionally compound proposition such as $K^m p^k$, for $m \geq k$ ('^{m}' indicates the order of the argument for K), is m+1.

The lowest order of an extensionally compound proposition is identical with the order of that its subproposition which has the highest order in it.

17. Russellian typing (typing rule for propositions - examples)

- (because of cumulativity, we speak about the lowest order, not about 'the' order; a proposition can have one order in one context and another order in another context)
- let '/' abbreviate '... has the (lowest) order ...'; here are some examples:

p^1 / 1; K^1p^1 / 2; $K^2K^1p^1$ / 3; K^2p^1 / 3 (cumulativity; p^1 serves here as a 2nd-order argument);

$(p^1 \wedge q^1)$ / 1; $(p^2 \wedge q^1)$ / 2; $K^2(p^2 \wedge q^1)$ / 3

- ill-formed formulas representing no propositions: 'K^1p^2', '$K^1K^2p^1$', '$K^1(p^2 \wedge q^1)$'

18. Preliminaries to Fitch’s knowability paradox

- untyped forms of claims

NonOmn $\exists p\ (p \wedge \neg Kp)$ // “there is an unknown truth”

Ver $\forall p\ (p \supset \Diamond Kp)$ // Verificationism, “every truth is knowable”

- FP is an inference which derives, by means of uncontroversial principles of epistemic logic, Omn(iscience) from Ver(ificationism) (this is paradoxical):

Omn $\forall p\ (p \supset Kp)$ // “every truth is known”

19. Crucial part of Fitch's knowability paradox and its blocking

1. $\mathrm{K}(p \wedge \neg \mathrm{K}p)$ // assumption derived from NonOmn and Ver
2. $(\mathrm{K}p \wedge K\neg \mathrm{K}p)$ // by Dist(tributivity) of K over $\wedge$
3. $(\mathrm{K}p \wedge \neg \mathrm{K}p)$ // by Fact(ivity) of knowledge, $\mathrm{K}p \vdash p$
4. $\neg \mathrm{K}(p \wedge \neg \mathrm{K}p)$ // *reductio*, since 3. is contradictory
5. $\Box\neg \mathrm{K}(p \wedge \neg \mathrm{K}p)$ // by Necessitation rule (if $\vdash p$, then $\vdash \Box p$)
6. $\neg\Diamond \mathrm{K}(p \wedge \neg \mathrm{K}p)$ // by Exchange rule for modal operators

---

1^{L}. $\mathrm{K}^2(p^1 \wedge \neg \mathrm{K}^1 p^1)$ // assumption

2^{L}. $(\mathrm{K}^2 p^1 \wedge \mathrm{K}^2\neg \mathrm{K}^1 p^1)$ // by the 2nd-order version of Dist

3^{L}. $(\mathrm{K}^2 p^1 \wedge \neg \mathrm{K}^1 p^1)$ // by the 2nd-order version of Fact

- 3^{L}. seems to be non-contradictory (*cf*. the discussion below)

Logika: systémový rámec rozvoje oboru v ČR a koncepce logických propedeutik pro mezioborová studia (reg. č. CZ.1.07/2.2.00/28.0216, OPVK)

20. Full (untyped) inference

1. $\exists p\ (p \wedge \neg \mathrm{K}p)$ // NonOmn(iscience)
2. $(p \wedge \neg \mathrm{K}p)$ // an instance of 1.
3. $\forall p\ (p \supset \Diamond \mathrm{K}p)$ // taking here Ver as "axiom"
4. $(p \wedge \neg \mathrm{K}p) \supset \Diamond \mathrm{K}(p \wedge \neg \mathrm{K}p)$ // substituting 3. for *p* in Ver
5. $\Diamond \mathrm{K}(p \wedge \neg \mathrm{K}p)$ // by MP on 4. and 3.
6. $\mathrm{K}(p \wedge \neg \mathrm{K}p)$ // assumption *per absurdum*
7. $(\mathrm{K}p \wedge K \neg \mathrm{K}p)$ // by Dist(tributivity) of K over $\wedge$
8. $(\mathrm{K}p \wedge \neg \mathrm{K}p)$ // by Fact(ivity) of Knowledge, $\mathrm{K}p \mid\!- p$
9. $\neg \mathrm{K}(p \wedge \neg \mathrm{K}p)$ // *reductio*, since 7. is contradictory
10. $\Box \neg \mathrm{K}(p \wedge \neg \mathrm{K}p)$ // by Necessitation rule (if $\mid\!- p$, then $\mid\!- \Box p$)
11. $\neg \Diamond \mathrm{K}(p \wedge \neg \mathrm{K}p)$ // by Exchange rule for modal operators

thus, 11. contradicts 5., i.e. adding Ver to NonOmn leads to a contradiction

21. Absence of solution in non-cumulative typing frameworks (Objection 4)

- note that the blocking the paradox can be provided even by Tarskian typing, provided it is cumulative (is it?)
- the *master argument* against typing by Carrara and Fassio (2011) in fact shows that there is a revenge form of the paradox which affects non-cumulative frameworks (recall that Churchian RTT is explicitly formulated as cumulative)
- having no cumulativity, the correct form of 2. would be

  $2.^{LCF}$ $(K^1p^1 \wedge K^2\neg K^1p^1)$,

  not $2.^{L}$ $(K^2p^1 \wedge K^2\neg K^1p^1)$; however, such $2.^{LCF}$ entails the contradictory

  $3.^{LCF}$ $(K^1p^1 \wedge \neg K^1p^1)$,

  thus the *reductio* would not be blocked, the critics say

22. Special semantic assumption for solving FP

- the proposition ($K^2p^1 \wedge \neg K^1p^1$) is not contradictory only provided the following rule (call it ‘*Epistemic rule*’ for the sake of our discussion) is *invalid*:

$$K^2p^1 \vdash K^1p^1$$

- note that the need of this *special assumption about K* was not foreseen by Church
- the invalidity of the Epistemic rule was discussed by Williamson (2000), Paseau (2008), Linsky (2009), and also Carrara and Fassio (2011)
- its discussion led to the recognition of logical / philosophical reasons for content of / epistemic access to propositions

23. Why the Epistemic rule is not valid

- the Epistemic rule says that if a proposition p^1 is known^2, it is already known^1 (on some bad understanding of cumulativity, this would be trivially true)
- my explanation of the invalidity echoes Paseau's (2008) claim that knowing^2 p^1 differs from knowing^1 p^1 because the higher-order knowledge^2 consists in knowing an *epistemic route* to p^1
- a possible objection: this is too strong condition
- a weaker condition is sufficient:

  a reasonable explication of knowledge of p preserves that p is justified, whereas being justified is defined (explicated) by:

  $(\text{Justified}^2\, p^1) \dashv\vdash \exists q^2 (\text{ReasonFor}^2\, q^2\, p^1)$

24. Why the Epistemic rule is not valid (epistemic access)

- note that q^2, which is a(n irreducibly a 2nd-order) reason for p^1, need not to be an information about an epistemic route to p^1 (e.g. "Xenia told me p^1 and Xenia is a reliable source"), though such route (a proposition) is an obvious candidate (for another example, consider the proposition "p^1 is a consequence of this and that proof", assuming that such claims have a meta-character)
- note that such consideration provides a *logical epistemic reason* for Russellian typing

25. Why the Epistemic rule is not valid (whole picture)

- thus the *whole picture of Russellian typing knowledge* is as follows
- if a proposition p^k is known, the epistemic proposition $\mathrm{K}^m p^k$ (for $m \geq k$) is typed because of the presence of K^m, which has a connection with that the epistemic attitude of knowing^m this p^k involves a justification by means of a certain q^m
- adding here also the *philosophical epistemic reason*: that the order of an epistemic proposition is higher than the order of a basic proposition consists in that the epistemic propositions have a ‘reflective’ character, thus they are ‘operating on’ basic factual propositions (compare “Fido is a dog” and “Xenia knows that Fido is a dog”)

26. The logical argument against Russellian typing (Objection 5)

- such criticism needs much more extensive debate (not here)
- sketched by Williamson (2000), elaborated by Carrara and Fassio (2011), and mainly by Hart (2009)
- evoking in fact Gödel’s criticism of Russell’s RTT (1944), they let RTT to speak about itself, allowing quantification over types, *cf.* e.g. the formula

  ‘$(p\wedge\forall t\neg\mathrm{K}^t p)$’
- they formulate a *revenge form of FP* for the approach
- however, the formulas are ill-formed, since they violate VCP; there is thus no revenge at all
- to quantify over types of some (object-)RTT, one must construct a meta-RTT for that object-RTT (a similar suggestion was made by Paseau 2008); but the appropriately modified revenge form of FP is easily blocked by typing

27. Summing up

- Russellian typing knowledge is a lively option for solving FP
- the critics wrongly confused it with Tarskian typing which is arguably an *ad hoc* approach
- the leading motive of Russellian typing is that propositions and intensional operators have a fine-grained intensional identity criteria, whereas their very formation is non-circular (VCP)
- we can find all four kind of justifications for this, viz. philosophical / logical reasons why to distinguish kinds (levels, types) of the content / epistemic access of knowledge
- Russellian typing knowledge is a part of a greater effort

  If, following early Russell, we hold that the object of assertion or belief is a proposition and then impose on propositions the strong conditions of identity which it requires, while at the same time undertaking to formulate a logic that will suffice for classical mathematics, we therefore find no alternative except for ramified type theory with axioms of reducibility. (Church 1984, 521)

References

Carrara, M., Fassio, D. (2011): Why Knowledge Should Not Be Typed: An Argument against the Type Solution to the Knowability Paradox. *Theoria* 2, No. 77, 180-193.

Giaretta, P. (2009): The Paradox of Knowability from a Russellian Perspective. *Prolegomena* 8, No. 2, 141-158.

Gödel, K. (1944): Russell's Mathematical Logic. In: Schilpp, P.A. (ed.), *The Philosophy of Bertrand Russell*, Evanston, Chicago: Northwestern University, 125-153.

Fitch, F. B. (1963): A Logical Analysis of Some Value Concepts. *The Journal of Symbolic Logic* 28, No. 2, 135-142.

Florio, S., Murzi, J. (2009): The Paradox of Idealization. *Analysis* 69, No. 3, 461-469.

Halbach, V. (2008): On a Side Effect of Solving Fitch's Paradox by Typing Knowledge. *Analysis* 68, No. 2, 114-120.

Halbach, V. (2011): *Axiomatic Theories of Truth*. Cambridge: Cambridge University Press.

Hart, W. D. (2009): Invincible Ignorance. In: Salerno, J. (ed.), *New Essays on the Knowability Paradox*. Oxford: Oxford University Press, 321-323.

Church, A. (1973-1974): Russellian Simple Type Theory. *Proceedings and Addresses of the American Philosophical Association* 47, 21-33.

Church, A. (1976): A Comparison of Russell's Resolution of the Semantical Antinomies with that of Tarski. *Journal of Symbolic Logic* 41, No. 4, 747-760.

Church, A. (2009): Referee Reports on Fitch's "A Definition of Value". In: Salerno, J. (ed.), *New Essays on the Knowability Paradox*. Oxford: Oxford University Press, 13-20.

Jago, M. (2010): Closure on Knowability. *Analysis* 70, No. 4, 648-659.

Linsky, B. (2009): Logical Types in Some Arguments about Knowability and Belief. In: Salerno, J. (ed.), *New Essays on the Knowability Paradox*. Oxford: Oxford University Press, 163-179.

Paseau, A. (2008): Fitch's Argument and Typing Knowledge. *Notre Dame Journal of Formal Logic* 49, No. 2, 153-176.

Paseau, A. (2009): How to Type: Reply to Halbach. *Analysis* 69, No. 2, 280-286.

Peressini, A. F. (1997): Cumulative versus Noncumulative Ramified Types. *Notre Dame Journal of Formal Logic* 38, No. 3, 385–397.

Raclavský, J. (2009): *Names and Descriptions: Logico-Semantical Investigations* (in Czech). Olomouc: Nakladatelství Olomouc.

Russell, B. (1903/2006): *Principles of Mathematics,* Cambridge: Cambridge University Press.

Russell, B. (1908): Mathematical Logic as Based on the Theory of Types. *American Journal of Mathematics* 30, No. 3, 222-262.

Tichý, P. (1988): *The Foundations of Frege's Logic*. Berlin, New York: Walter de Gruyter.

Williamson, T. (2000): *Knowledge and its Limits*. Oxford: Oxford University Press.

Whitehead, A. N., Russell, B. (1910-1913): *Principia Mathematica*. Cambridge: Cambridge University Press.