Russell's Propositional Functions from the Viewpoint of Tichý's Type Theory

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Logika: systémový rámec rozvoje oboru v ČR a koncepce logických propedeutik pro mezioborová studia (reg. č. CZ.1.07/2.2.00/28.0216, OPVK)

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Abstract

We investigate crucial notion of the logical system of Principia Mathematica from the viewpoint of Pavel Tichý's ramified theory of types. It is important to say that Tichý's ramified type theory involves a simple type theory in its bottom part, thus it is rather richer than Russell's type theory. Assuming that propositional constructions are comparable with Tichý's so-called constructions (which are, roughly, algorithms), we discover a number of surprising dissimilarities between the two systems. We will see that, unlike constructions, propositional functions are too simple in the sense that they have no connection to set-theoretical entities. This deprived Russell's system of certain richness which could be interesting e.g. for mathematicians.

I. Introduction

I. Propositional functions (PFs)

A propositional function is nothing, but, like the most of the things one wants to talk in logic, it does not lose its importance through that fact

(Russell 1918/1998, 64; The Philosophy of Logical Atomism)

– basic characteristics: *propositional function* is an entity which becomes a proposition if its undetermined constituents (variables) become determined (some values are assigned to those variables)

I. What are propositional functions?

- not only E. Mares (2011, Propositional Functions, SEP) asks what PFs in Russell's work are
- the answer depends on how one understands their constituents
- let their nature constitute the content of *the question A.*

- in the present talk I will be interested rather in another question, *the question B.*: what would happen if PFs were of such and such nature; in other words, I will discuss the character of Russell's *logical theory*

I. Two extreme views on PFs

- two possible, yet a bit extreme answers to the question A.
- *realistic interpretation*: as already in The Principles of Mathematics (POM), PFs are made from individuals, variable individuals, properties, etc., thus they are 'real' entities

Propositional functions, accordingly, are universals for Russell in his 1910-13 PM ontology, and as such they may also be called properties and relations (in intension). (Cocchiarella 1989, 47)

- *nominalistic interpretation*: PFs are concatenation of letters

Variables, in the easiest sense, are letters; and what contain them are notational expressions. ... propositions, unlike the individuals, are evidently notations; at any rate they contain variables ... his functions also are notational in character; they seem simply to be open sentences, sentences with free variables (Quine 1967, 151)

I. Contemporary view on PFs

- *linguistic interpretation*: PFs are linguistic entities whereas 'linguistic' does not mean 'without a meaning' as nominalist would maintain, but rather that it relates to our language as a human capability
- *cf.* mainly G. Landini but see also B. Linsky (and others)

Russell's discussion of propositions or propositional functions as linguistic items cannot be read in the narrowly syntactic fashion of a nominalist. Russell always considered language to be what we would call "interpreted", a piece of language with the meaning it has. ... When Russell says that an entity is "linguistic", he means that it is the subject matter of logic, and not an ordinary concrete particular. ... It is not the uninterpreted syntax that the nominalists would make of it. (Linsky 1999, 15)

- it is confirmed by the late Russell (My Philosophical Development)

I no longer think that the laws of logic are laws of things; on the contrary, I now regard them as purely linguistic (MPD, 102)

I. PFs as Tichý's constructions

- despite that I subscribe to the linguistic interpretation, in this talk I will construe PFs rather as abstract entities because I want to compare them with Pavel Tichý's so-called *constructions* which are, unambiguously, abstract, extra-linguistic entities
- such comparison can put a light on Russell's logical theory (the B. question)

- preliminary characteristics of Tichý's constructions: extra-linguistic structured entities akin to algorithms; they consist of entities of set-theoretical objects such as individuals or mappings

I. Content of the presentation

I. From Principles (POM) to *Principia* (PM)

II. Tichý's early logical system and the notion of construction

III. PFs as constructions

Typical ambiguity

Variables

Vicious circle principle(s) (VCPs) and typing

IV. Ramified theory of types (RTT)

Predicativity

Axiom of Reducibility

V. Some conclusions

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I. Towards PFs of PM (1/3)

- in POM (1903), Russell thought of propositions as structured entities consisting of individuals, possibly 'variable-individuals' (= objectual notion of variable), and structured attributes (properties or relations), or even classes (which are still allowed)

  Words all have meanings, in the simple sense that they are symbols which stand for something other than themselves. But a proposition … does not itself contain words: it contains the entities indicated by words. (POM, 47)

- *propositional functions* are like propositions, except containing variables instead of concrete individuals or attributes, etc. (= *objectual notion of PFs*)
- 2 paradoxes of POM (*cf.* Appendix B.): *Russell's paradox* (solved by Russell's simple *type theory*) and Russell's *paradox of propositions*; thus classes and propositions are problematic entities

I. Towards PFs of PM (2/3)

- Russell estimated that the paradox of propositions could be solved by splitting the type of propositions into partial types (later called 'orders'), which is, however, a 'harsh and highly artificial' suggestion (POM, 528)
- Russell did extensive investigations to find a satisfiable way out from paradoxes which would, which is very important, enable to establish logic as the most universal science
- complicated development through *zig-zag theory*, *limitation of size theory* and *no-class theory* as theories of classes
- while adopting *substitutional theory* and then *ramified theory of types* as the logical theories
- (*cf.* e.g. Landini, Linsky, de Rouilhan 2004)

I. Towards PFs of PM (3/3)

- each theory was chosen as ‘smaller evil’, being a replacement of a more unsatisfactory theory
- in On Some Difficulties in the Theory of Transfinite Numbers and Order Types Russell aptly said that

a. the disputable notion of PF has the same problems as the notion of class

the postulate of the existence of classes and relations is exposed to the same arguments, *pro* and *con*, as the existence of propositional functions as separable entities distinct from all their values. (Russell 1907, 45)

b. but the notion of class, which leads to paradoxes, is better to be abandoned at all (= no-class theory)

From further investigation I now feel hardly any doubt that the no-classes theory affords the complete solution of all the difficulties stated in the first section of this paper. (Russell 1907, 58)

I. MLBT and PM (1/3)

- the paper Mathematical Logic as Based on the Theory of Types (MLBT, 1908) is Russell's first complete attempt to offer both no-class theory as the ontological view and ramified theory of types as its logical theory
- essentials of MLBT are reproduced in PM (*Principia Mathematica*, Whitehead and Russell 1910-13); there are several minor differences between MLBT and PM (difference in type theories, reminders of substitutional theory in MLBT, etc.) which will be ignored here
- accepted entities: individuals, PFs, propositions, ...(and?)
- classes are only represented by PFs (PM 24), i.e. they are excluded from Russell's own ontology
- attributes can also be represented by PFs, but they seem still present

I. MLBT and PM (2/3)

- in MLBT Russell aptly says (p. 262) that he offers his theory as a mathematical theory but without a philosophical discussion which must be offered separately
- the policy was preserved also in PM because the system of PM is intended to be as universally applicable as possible
- in a consequence of this, some philosophical questions cannot be convincingly resolved, we simply lack any textual evidence
- for instance, attributes seem to be still tolerated in PM:

  A 'property of *x*' may be defined as a propositional function satisfied by *x*. (PM, 166)

  because Russell does not repudiate them as classes (class terms are incomplete symbols, 144, while properties/relations or their names do not)

I. MLBT and PM (3/3)

- for a better example, existence of propositions either in or behind PM's ontology is a controversial matter: same say that they are abandoned (e.g. Landini 1996, 1998; Hylton 2008), while others (e.g. Church 1976, 748) do not
- its is common to explain the omission of propositions as a consequence of Russell's multiple relation theory of judgement; yet both crucial passages do not really repudiate propositions:

A proposition is not a single entity, but a relation of several; hence a statement in which a proposition appears as subject will only be significant if it can be reduced to a statement about the terms which appear in the proposition. (PM, 51)

a "proposition," in the sense in which a proposition is supposed to be *the* object of judgement, is a false abstraction, because judgement has several objects not one ... the phrase which expresses a proposition is what we call an "incomplete" symbol; it does not have meaning in itself, but requires some supplementation in order to acquire a complete meaning (PM, 46)

II. Tichý's logical notions

II. Tichý's objectual approach to logic

- Pavel Tichý (*1936 Brno, #1994 Dunedin)
- key book: The Foundation of Frege's Logic (1988, de Gruyter)
- *objectual conception of logic*: to study properties of (abstract) logical entities, rather than peculiar logical notations

Turning logic into the study of an artificial language (which nobody speaks) does not strike me as the height of wisdom. A formula of symbolic logic, just like a piece of musical notation, is utterly uninteresting in its own right. Its interest stems exclusively from its ability to represent something other than itself. (Tichý 1988, Preface)

Both Frege and Russell took, ..., an objectual view of logic. They both devised and *used* ingenious symbolic languages, whose various modifications were to become the stock in trade of symbolic logic. Yet they themselves were not symbolic logicians; a symbolism to them was not the *subject matter* of their theorizing but a mere shorthand facilitating discussion of extra-linguistic entities. (Tichý 1988, Preface)

II. Tichý's attitude to Russell's logic

- in his 1988 book, Tichý studied work of Frege and his followers, not Russell
- his reactions to Russell's thoughts are rare (except criticism of his theory of descriptions) and some of them seems to be less convincing
- on the other hand, Tichý provided a ramification of his early simple theory of types and insisted on its relation to Russell's RTT

It is one of the aims of the present work to propose a non-linguistic theory of the variable and to give a consistently objectual version of Russell's Ramified Theory of Types. I will argue that the 'hierarchy of entities' which results from this rectification of Russell's system is not only a useful tool for diagnosing the flaws and ambiguities in Frege's logic but also the right medium for modelling our whole conceptual scheme. (Tichý 1988, Preface)

II. Tichý's Church-like Simple Theory of Types

- Tichý's type theory (TTT) combines both simple (STT) with a feature of common ramified theory of types (RTT)
- within its STT part, Tichý's does classify *functions in the extensional sense* (mappings), which all were abandoned by Russell
- the STT offered by Tichý already in early 1970s is a generalization of the classical (Church 1940):

Let *B* (basis) be a set of pair-wise disjoint collections of objects:

a. Every member of *B* is an (atomic) *type over B*.

b. If ξ, ξ_1, ..., ξ_n are types over *B*, then $(\xi\xi_1...\xi_n)$, i.e. collection of total and partial functions from ξ_1, ..., ξ_n to ξ, is a (molecular) *type over B*.

c. Nothing is a *type over B* unless it so follows from a.-b.

II. Early Tichý's logic (1/2)

- early Tichý's logic, e.g. (An Approach to Intensional Logic, 1971, Nous), later labelled '*Transparent Intensional Logic*', used types of individuals, truth-values, possible worlds, and real numbers/time-moments as atomic types (*PWS-intensions* by Tichý are total or partial functions from possible worlds and moments of times)
- Tichý soon developed a *deduction system* for it (unpublished book 1976, selection in 1982, 1986) and provided a number of *semantical analyses* (modalities, propositional attitudes, intensional transitives, temporal adjectives, verb tenses, episodic verbs, semantics of descriptions) and *explicated* several *philosophical notions* (e.g. truthlikeness, fact, truth, ...), *cf.* (Tichý 2004 - collected papers) and also works of Tichý's followers

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II. Early Tichý's logic (2/2)

- the language of his 1971-paper is a λ-notation
- but Tichý immediately realized that the meanings of (NL-)expressions cannot be intensions or extensions which are too coarse grained; one needs rather certain structured entities with a fine-grained structure (these are called *hyperintensions* nowadays; *cf.* the usual arguments in their favour)
- Tichý began to think of *objectual correlates of λ-terms* as structured entities having algorithmic nature ('intensional understanding of λ-terms')
- (Tichý thus connected his 1971 theory with his 1969 theory – Intensions in Terms of Turing Machines, Studia Logica – where he proposed that meanings of expressions are algorithms)
- Tichý called that objects *constructions* (having a geometrical inspiration)

II. Constructions (1/4)

- constructions are structured abstract, extra-linguistic procedures (algorithms); they are usually recorded by means of λ-terms
- any object *O* is constructible by infinitely many *equivalent*, yet *not identical*, constructions (= *intensional principle of individuation*)
- each construction *C* is specified by two features:
  i. *which* object *O* (if any) is constructed by *C*
  ii. *how C* constructs *O* (by means of which subconstructions)
- for a philosophical defence of the notion see esp. (Tichý 1988, 1986)
- *semantic scheme*: expression -> construction (meaning) -> intension/non-intension (denotation); (reference in *W* and *T*: the value of intension in *W*, *T*)

II. Constructions - kinds of (2/4)

– for exact specification of constructions see (Tichý 1988)

– six kinds of constructions, two of them being omitted here

(where X is any object or construction and C_i is any construction):

a. variables	x	(not as letters!)
b. trivializations	${}^0X$	('constants')
c. compositions	$[C\ C_1...C_n]$	('applications')
d. closures	λxC	('λ-abstractions')

– definition of subconstructions, free/bound variables, etc.

– constructions v-constructing nothing (v is valuation) are also allowed

- TIL-λ-terms are only used to denote constructions ('fixed interpretation')

II. Constructions - examples (3/4)

- example of equivalent constructions; the following function (call it ‘XYZ’):

  $1 \rightarrow -2$

  $2 \rightarrow 1$

  $3 \rightarrow 6$

  :   :

  is *v*-constructed by infinitely many constructions (of various kinds), e.g.:

    $\lambda n\ [[n\ {}^0\!\times\ n]\ {}^0\!-\ {}^0 3]$

    $\lambda n\ [\ [n\ {}^0\!+\ [{}^0\text{SquareOf}\ n]]\ {}^0\!-\ [{}^0 3\ {}^0\!+\ n]\ ]$

    ^{0}XYZ (the trivialization of “XYZ” directly constructs “XYZ”)

    [^{0}IdentityFuntion ^{0}XYZ]

- thus constructions echoes *functions in intensional sense*

II. Constructions - examples (4/4)

- let us put a closer look on 'behaviour' of construction, their *v*-constructing:

  $[{}^0 2\ {}^0{+}\ n]$ *v*-constructs a number dependently on *v* in several steps

  - ${}^0 2$ constructs (for any *v*) the number 2, ${}^0{+}$ constructs (for any *v*) the mapping +
  - *n* *v*-constructs the number *N* which is assigned to *n* by *v*
  - $[{}^0 2\ {}^0{+}\ n]$ *v*-constructs the number *M* which is a result of applying + to <2,*N*>
  - ('$[{}^0 2\ {}^0{+}\ n]$' stands for the procedure-construction, not for its result *M*)

  $\lambda n[{}^0 2\ {}^0{+}\ n]$ constructs (for any *v*) a mapping from numbers to the appropriate results of $[{}^0 2\ {}^0{+}\ n]$; ('$\lambda n[{}^0 2\ {}^0{+}\ n]$' stands for the procedure)

III. Comparing Russell's functions with Tichý's constructions

III. Constructions vs. PFs

- in his 1988 book, Tichý seems to occasionally suggest that Russell's propositional functions (attributes) are similar to (some) his constructions
- this is wrong for the following reasons:
    i. constructions are strictly extralinguistic objects, while PFs are not
    ii. constructions construct objects, while PFs do not
    iii. PFs are not isomorphic to λ-terms (a controversial issue, *cf.* Church, Klement, Hylton, Landini)
- *dissimilarities* i. and iii. can be perhaps ignored
- however, the *dissimilarity ii. is crucial* and has many consequences
- on the other hand, there are *noteworthy similarities* between the two systems (*cf.* esp. the intensional individuation of constructions and PFs)

III. Typical ambiguity and PFs (1/3)

- to understand Russell's PFs, one should understand his construal of functions; Russell, however, has no ultimate answer on what functions are:

  The question as to the nature of a function is a by no means an easy one. It would seem, however, that the essential characteristic of a function is *ambiguity*. (PM, 41)

- he explains that φx ambiguously denotes a value of the function, *cf.* with φa which unambiguously denotes the value of the function
- (typical ambiguity was often wrongly understood as Curry-like type-polymorphism)
- compare it with Tichý's notion of constructions: the way $\lambda x \varphi x$ constructs its value is quite clear from the specification of constructions and kinds of their forming

III. Typical ambiguity and PFs (2/3)

- Russell's PFs should be perhaps understood in terms of substitution; this is, however, unclear (*cf*. Gödel 1944)
- in contrast with this, the way the construction $\lambda x \varphi x$ can be converted to the propositional construction φa is clear from the definition of substitution (and related notions) provided mainly in (Tichý 1982, 1986, 1988)

- in Tichý's system, the nature of a propositional construction is clear also from the PWS-intension (which have truth-values as its values) constructed by it
- in Russell's system, something of this sort is impossible (there are even no truth-values in PM)

III. Variables (1/2)

- understanding variables is necessary for our understanding of PFs as well
- Russell did not provide an explanation of the notion, he gave only a hint

In mathematical logic, any symbol whose meaning is not determinate is called a *variable*, and the various determinations of which its meaning is susceptible are called the *values* of the variable. (PM, 4)

- Tichý argued (1988, chap. 4) that Russell cannot explain 'coordination of values' of variables such as x_m and x_n and thus their real identity
- compare with Tichý's objectual notion of variable (*ibid.*): variables (the constructions) construct dependently on valuation; valuation are sequences (precisely: fields) of values; given v, x_m v-constructs the m-th member of v, while x_m v-constructs its n-th member

III. Variables (2/2)

- Tichý on Russell's and Frege's notion of variable:

  Russell's logic suffers from ambivalence no less than Frege's does. ... the ambivalence has the same main source: a failure to devise a viable objectual account of the variable. It was this failure which forced both authors to deviate in crucial points from their own objectual approach and to resort to linguistic ascent. ... It is one of the aims of the present work to propose a non-linguistic theory of the variable (Tichý 1988, Preface)

- in PM, Russell repeatedly speaks about denoting variables by letters, saying also that they are symbols and notoriously confusing them with objects they stand for: individuals, function etc. (what are Russell's variables?)

  function which involves no variables except individuals (PM, 4)

  The variables occurring in the present work, from this point onward, will all be either individuals or matrices (PM, 164)

III. VCPs and typing (1/5)

- the relation of PFs (which contain variables), variables, and the famous Vicious Circle Principle (VCP) as well as typing is best explained as follows:

  The division of objects into types is necessitated by the vicious-circle fallacies which otherwise arise. These fallacies show that there must be no totalities which, if legitimate, would contain members defined in terms of themselves. Hence any expression containing an apparent variable must not be in the range of that variable, *i.e.* must belong to a different type. Thus the apparent variables contained or presupposed in an expression are what determines its type. This is the guiding principle in what follows. (PM, 168)

- in other words, ranges of variables bring about ramification within the type theory

III. VCPs and typing (2/5)

- the third sentence ('Hence any expression ...') is offered in MLBT (237) as the *technical formulation* of VCP, which thus corrects the formulation from (MLBT, 225):

  Whatever contains an apparent variable must be of a different type from the possible values of that variable; we will say that it is of a *higher* type.

- in PM, however, the only announced formulation of VCP is the formulation from (MLBT, 225):

  Whatever involves *all* of a collection must not be one of the collection"; or, conversely: "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total (PM, 41)

- (Gödel 1944 is known for an examination of Russell's non-technical formulations)

III. VCPs and typing (3/5)

– note that VCP(s) can be understood as a consequence(s) of the much more fundamental principle, viz. the *Principle of Specification* (Raclavský 2009):

one cannot specify an item by means of the item itself,

which is inspired and generalized from:

> a function is not a well-defined function unless all its values are already well-defined. It follows from this that (PM, 41)

and:

> Non-circularity is a constraint governing the formation of wholes of any sort. In assembling a car, for example, one can hardly use anything which depends on the very same car's being assembled already. (Tichý 1988, 47)

III. VCPs and typing (4/5)

- compare Russell's VCP with four VPCs valid in Tichý's TT (Raclavský 2009):

i. *Constructional VCP*: no construction can (*v*-)construct itself

- e.g., a variable *c* which *v*-constructs constructions cannot be in its own range, it cannot *v*-construct itself; quite analogously for [...*c*...], etc., but also for (say) ^0C

- Russell's technical definition of VCP covers only a partial case: constructions/PFs involving variables

- as a consequence, cumulativity seems to be missing in Russell's RTT (Church 1976, 747); *cumulativity* means that every *k*-order constructions is a *k*+1-order construction; (in PM there is no reason to increase order if it is not necessary, e.g. in the case of ^0C)

III. VCPs and typing (5/5)

- in STT which admits only extensional functions (mappings), there is implemented:

ii. *Functional VCP*: no function (mapping) can contain itself among its own arguments or values

- (as in other VCPs: the function *F* cannot contain a function *G* presupposed by *F*; cf. with Russell's explanation of presupposing at PM, 42)

- i. entails iii. *Constructional-Functional VCP*: no construction *C* can (*v*-)construct a function having *C* among its own arguments or values
- ii. entails iv. *Functional-Constructional VCP*: no function *F* can contain a construction of *F* among its own arguments or values

IV. Comparing Russell's type theory with Tichý's

IV. Ramified type theory (1/5)

- VCP leads to type theory
- in PM, several similar definitions of TT are offered (e.g. the version on 169-172 uses predicativity);
- here is a proposition-version from MLBT:

First type contains *individuals;*

Second type contains *first-order propositions*, i.e. a. elementary propositions (not involving variables) and b. propositions whose only apparent variables being individuals;

Third type contains *second-order propositions*, i.e. propositions having variables for first-order propositions as its apparent variables;

Etc.

IV. Ramified type theory (2/5)

- Tichý is right in saying that Russell's RTT 'is a rather drastically pruned version thereof' (1988, 68)
- Tichý's RTT contains his early STT and STT is also used in higher-order

Let *B* a base.

1. **(t_1i)** Every member of *B* is a *type of order* 1 *over B*.

   ($t_1$ii) If $0<m$ and ξ, ξ_1, ..., ξ_m are any *types of order* 1 *over B*, then the collection $(\xi\xi_1...\xi_m)$ of all *m*-ary (total or partial) functions from ξ_1, ..., ξ_m into ξ is also a *type of order* 1 *over B*.

   ($t_1$iii) Nothing is a *type of order* 1 *over B*, unless it so follows from (t_1i) and ($t_1$ii).

IV. Ramified type theory (3/5)

- ramification is made by stratification of constructions ...

2. **(c_ni)** Let ζ be any *type of order n over B*. Every variable ranging over ζ is a *construction of order n over B*. If X is of (i.e. belongs to) type ζ, then ^0X is a *construction of order n over B*.

   ($c_n$ii) If $0<m$ and C, C_1, ..., C_m are *constructions of order n over B*, then $[C\ C_1...C_m]$ is a *construction of order n over B*. If $0<m$, ζ is a type of order n over B, and C, as well as the distinct variables x_1, ..., x_m, are *constructions of order n over* B, then $[\lambda x_1...x_m C]$ is a *construction of order n over B*.

   ($c_n$iii) Nothing is a *construction of order n over B*, unless it so follows from (c_ni) and ($c_n$ii).

IV. Ramified type theory (4/5)

- ... and also by stratification of functions-mappings to or from constructions (note the special cumulativity formulated in (t_{n+1}i))

Let $*_n$ be the collection of *constructions of order n over B*. The collection of *types of order n+1 over B* is defined as follows:

(t_{n+1}i) $*_n$ and every *type of order n* is a *type of order n*+1 *over B*.

(t_{n+1}ii) If $0<m$ and $\xi, \xi_1, ..., \xi_m$ are *types of order n*+1 *over B*, then the collection $(\xi\xi_1 ... \xi_m)$ of all *m*-ary (total and partial) functions from $\xi_1, ..., \xi_m$ into ξ is also a *type of order n*+1 *over B*.

(t_{n+1}iii) Nothing is a *type of order n*+1 *over B*, unless it so follows from (t_{n+1}i) and (t_{n+1}ii).

IV. Ramified type theory (5/5)

- the whole system of PM was criticized by F.P. Ramsey (1925) whose criticism was repeated by Quine and others;
- the basic Ramsey's idea was to favour exclusively extensional functions, thus the whole PM's ramification seems unnecessary
- but Ramsey clearly realized the price: incapability to solve 'extra-logical' paradoxes
- this shows why such approach had, in Russell's eyes, an irretrievable drawback: the resulting system would be only a partial attempt to reconstruct our *whole* conceptual sphere, such logic is far from being universal
- both Russell and Tichý were led to RTT by the principal insufficiency of STT

IV. Predicativity (1/3)

- a common definition: a definition (structured attribute,...) is *impredicative* iff it defines an object by means of quantifying over the totality the defined object belongs to
- note that Russell's notion of predicativity was a bit distinct: the (defining) predicative function is of the next order above its arguments (PM, 56)

- examples of impredicative attributes: "the only property *f* such that *f* is held by such and such individuals", "the only *n* such that for all *n*, ...*n*..." (math!), (all are predicative functions due to Russell)

IV. Predicativity (2/3)

- critics of impredicativity include H. Poincaré, H. Weyl (recently *cf.* e.g. S. Feferman 2005), their rationale seems to be a violation of VCP
- critics of (the alleged) Russell's dismissal of impredicativity include F. P. Ramsey (1931; repeated by Quine 1967), K. Gödel (1944); they referred to concrete or commonly used unproblematical examples (average Yaleman, a number of mathematical definitions - math. induction!)
- however, Ramsey's case (Quine's 'the average Yaleman') is a bit unpersuasive, because Russell's predicativity does not concern definitions of mere individuals but of structured complexes such as attributes (e.g. definition of "being the average Yaleman"), or propositions, and other intensional entities

IV. Predicativity (3/3)

- within TTT, an impredicative definition is impossible if it defines an entity which is (natively) an object of the same order as the definiens:
- let the definition be an object of order *k* (so it is an construction of order *k*-1); now, there are cases (suppose that the objects are natively of that order *k*): a. *k*-1-order construction (i.e. an object of order *k*), b. a function which is an object of order *k* (e.g. a function to *k*-1-order constructions), c. other object of order *k* (e.g. individuals)
- the only RTT-related possibility is a., while for Russell the only definable objects were individuals and propositional functions (or propositions); in other words, Russell’s reductionism is present also here

IV. Axiom of Reducibility (1/2)

- (in)famous Axiom of Reducibility (AR, Axiom of Classes) says that for every higher-order impredicative function there is a lower-order predicative one (PM, 59, 174)
- Russell could not prove this principle (cf. PM 60), considering it being far from self-evident (61); Russell even repudiated AR in PMII (1925)
- yet the principle is quite evident if one accepts, as Tichý did, extensional objects: it is a trivial fact that one an the same objects is constructed by two equivalent constructions *C* and *D* such that *D* is a impredicative higher-order construction and *C* is a predicative lower-order one

IV. Axiom of Reducibility (2/2)

- Russell wrongly justifies AR by means of the Napoleon example (Napoleon has all properties of great generals, PM 59)
- to quantify over 'all' properties is better provided by means of quantification over the highest possible order (the properties of even higher orders are surely artificial), quantification over first-order properties (most of them being equivalent to some higher-order impredicative ones) is always only a very partial attempt (=reductionism)
- the real importance of AR is made rather by informing us that there is a plenitude of (usually unnamed) first-order attributes which are equivalent to some higher-order (impredicative) ones

V. Some conclusions

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- Russell's logical project is very reductionistic: he excluded mainly classes which deprived him of the possibility to prove the Axiom of Reducibility and even equivalence of PFs of the same order is nearly unexplainable (Russell abandoned also descriptive functions, PM 33, which could correspond to descriptive terms)
- without extensional objects, Russell completely excluded STT from his theoretical background; this was rather criticized because a substantial part of mathematics treats extensional objects (classes being the best example); what is more important, exclusion of extensional objects led to

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Addendum

By a "propositional function" we mean something which contains a variable *x*, and expresses a *proposition* as soon as a value is assigned to *x*. That is to say, it differs from a proposition solely by the fact it is ambiguous: it contains a variable of which it the value is unassigned.

(PM, 41)

Propositional functions. Let *φx* be a statement containing a variable *x* and such that it becomes a proposition when *x* is given any fixed determined meaning. The *φx* is called "propositional function"; it is not a proposition, since owing to the ambiguity of *x* it really makes no assertion at all. (PM, 15)

A *propositional function* is simply *any expression containing an undetermined constituent*, or *several undetermined constituents, and becoming a proposition as soon as the undetermined constituents become* determined. If I say '*x* is a man' or '*n* is a number', that is a propositional function; so is any formula of algebra, say $(x+y)(x-y) = x^2 - y^2$. A propositional function is nothing, but, like the most of the things one wants to talk in logic, it does not lose its importance though that fact

(Russell 1918/1998, 64)

A "propositional function," in fact, is an expression containing one or more undetermined constituents, such that, when values are assigned to these constituents, the expression becomes a proposition. In other words, it is a function whose values are propositions.

(IMP, 155-156)

a propositional function is nothing but an expression. It does not, by itself, represent anything. But it can form a part of a sentence which does say something, true or false

(MPD, 69)

Whitehead and I thought of a propositional function as an expression containing an undermined variable and becoming an ordinary sentence as soon as a value is assigned to the variable (MPD, 124)