The Barber Paradox: on its Paradoxicality and its Relationship to Russell’s Paradox

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Logika: systémový rámec rozvoje oboru v ČR a koncepce logických propedeutik pro mezioborová studia (reg. č. CZ.1.07/2.2.00/28.0216, OPVK)

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Abstract

The Barber paradox is often introduced as a popular version of Russell's paradox, though some experts have denied their similarity, even calling the Barber paradox a pseudoparadox. In the first part of the talk, I am going to demonstrate mainly that in the standard (Quinean) definition of a paradox the Barber paradox is a clear-cut example of a non-paradox. Despite some outward similarities, it differs radically from Russell's paradox. I will also expose many other differences. In the second part of the talk, I will examine a probable source of the paradoxicality of the Barber Paradox, which is found in a certain ambivalence in terms of meaning. The two different readings of the crucial phrase yield distinct existential assumptions which produce the paradoxical conclusion.

Content

I. Quine's standard notion of paradox

I.1 Quine's standard notion of paradox (1/2)

- Quine (1966) 'The Ways of Paradox'
- *paradox* is an *argument* whose conclusion *contradicts* ('para-') one of its (possibly implicit) premises, which is a naïve theory ('doxa')
- Sainsbury put Quine's thought into this form:

  "an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises" (1987, 1, 'Paradoxes')

- Lycan (2010): argument is only an inconsistent set of propositions
- (paradox as an 2D-inference in Frege-Tichý sense)

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I.1 Quine's standard notion of paradox (2/2)

- a *solution to a paradox* consists either in a justified refutation of the problematic premise (naïve theory) or in a justified refutation of some derivation step
- for instance, consider the Liar paradox incorporating the naïve theory of truth and various solution rejecting either it or some derivation rule

I.2 Russell's paradox (RP)

- the naïve theory of RP is *naïve theory of sets* which is formulated here in a form of *unrestricted Axiom of Comprehension*:

$$\forall F\, \exists s\, \forall x\, (\, (x \in s) \leftrightarrow F(x)\,)$$

(in words, for any condition/formula *F* there exists a class *s* containing just and only those individuals *x*s who satisfy the condition *F*)

- Russell (1903) attempted to define class R with help of condition $(s \notin s)$

$$\mathrm{R} = \{\, s \mid s \notin s \,\}$$

the set of all and only those sets which are not members of themselves

I.3 The Barber paradox (BP)

- *the individual who shaves all and only those individuals who do not shave themselves*

$$\forall y\ (\ \mathrm{Shave}(x,y) \leftrightarrow \neg\mathrm{Shave}(y,y)\ )$$

- versions: catalogue of catalogues (F. Gonseth 1936); bibliography of bibliographies; secretaries of clubs C (Johnston 1940); Selbstmürder, ...)
- obviously, no individual can both *R* and non-*R* to itself, Thomson (1962):

$$\vdash \neg\exists x\ \forall y\ (\ R(x,y) \leftrightarrow \neg R(y,y)\ )$$

I.4 The Barber paradox (BP) – a note on its origin

- according to Alonzo Church (1963 in review of Johann Mokre 1952), the probable author of the BP is Ernst Mally
- Russell clearly rejected the BP as an analogy to RP:

"That contradiction [i.e. RP] is extremely interesting. You can modify its form; some forms of modification are valid and some are not. I once had a form suggested to me which was not valid, namely the question whether the barber shaves himself or not. You can define the barber as "one who shaves all those, and those only, who do not shave themselves". The question is, does the barber shave himself? In this form the contradiction is not very difficult to solve. But in our previous form I think it is clear that you can only get around it by observing that the whole question whether a class is or is not a member of itself is nonsense" (1918-1919/2010, 101; 'The Philosophy of Logical Atomism')

II. Russell's paradox and the Barber Paradox: similarities and dissimilarities

II.1 Dissimilarities between RP and the BP

- seeming similarity of crucial phrases ("the only entity such ... if and only if not ...")

- *dissimilarity*: the main phrase of the BP specifies an *empty set*,
  while the main phrase of RP specifies *no set*
- *dissimilarity* (Quine 1966, 12)*:* Russell's set *should* exist, but it does not;
  on the other hand, there is no surprise that the alleged barber does not exist (we will return to the problems of existence later)
- *main dissimilarity*: RP leads us to the refutation of naïve set theory (unrestricted Axiom of Comprehension),
  while the BP leads to the refutation of no naïve theory

I.2 Crucial dissimilarity between RP and the BP

- *crucial dissimilarity:* RP contains (as its premise) naïve theory of sets,
  while the BP contains *no naïve theory*,
- *hence*, by the standard Quinean definition of paradox, *the BP is not paradox at all*

- thus, no surprise that the BP is called *pseudoparadox* (Church 1940)
- it is correct that many rejected similarities between RP and the BP (Russell 1918-1919, Grelling 1936, ...)

I.3 A wrong similarity (inclusion vs. membership)

- in introductory math sources (cf. e.g. Perelman 1936, Gonseth 1936, ..., Joyce 2002), but even among some theoreticians of paradoxes (Rescher 2001) there is a tendency to understand the BP as an analogy to RP
- one reason is a confusion of $\in$ (..is member of ...) and $\subseteq$ (...contains/includes ...), whereas it holds:

$$|\!- \neg\exists x\, \forall y\, ((x\subseteq y) \leftrightarrow \neg(y\subseteq y))$$

which is analogous to (Thomson's):

$$|\!- \neg\exists x\, \forall y\, (R(x,y) \leftrightarrow \neg R(y,y))$$

III. Degrees of paradoxicality
and
the proper source of the Barber Paradox paradoxicality

III.1 Paradoxicality as a measure

- the second part of Quine’s notion of paradox is that *paradoxicality* comes in degrees (and that it is subjective):

  “One man's antinomy can be another man's veridical paradox, and one man's veridical paradox can be another man's platitude” (Quine 1966, 12)

- let $P(p)$ be paradoxicality of a paradox p:

  P(Horned Man p.) < P(RP) < P(Zwicker’s Hypergame p.)

- the BP can be turned into a proper paradox similar to the Horned Man paradox if one adds the premise:

  “Every property has always an instance.”

III.2 The proper source of the BP's paradoxicality: meaning ambiguity

- in ordinary understanding, the BP is a proper paradox because one starts with the assumption that and such barber can exists but one then finds that it cannot, which is a contradiction
- the proper source of the BP's paradoxicality lies in a *hidden switch of meaning* of the crucial phrase ("the only individual *x* which shaves ..."):

a) on *reflexive reading*, *x* shaves everybody (such and such), including himself

b) on *irreflexive reading*, *x* shaves everybody (such and such), excluding himself, i.e. the meaning of contains $\wedge(x \neq y)$

- the two readings have distinct existential consequences

III.3 The proper source of the BP's paradoxicality (reflexive reading)

(P) "There exists a barber that shaves all and only those who do not shave themselves."

(C) "There does not exist a barber that shaves all and only those who do not shave themselves."

- if (P) is contradictory to (C), it is read in *reflexive sense*
- since (C) is logical truth, (P) is a *logical contradiction*,
- thus *no such barber can possibly exist*

III.4 The proper source of the BP's paradoxicality (irreflexive reading)

- the reflexive reading is improbable, because one naturally starts with an assumption that *such barber can exist* by an contingent chance
- thus (P) must be read in *irreflexive sense* as in fact (P')

(P') "There exists a barber that shaves all and only those others who do not shave themselves."

(C') "There does not exist a barber that shaves all and only those who do not shave themselves."

- but the appropriate contradictory conclusion is (C'), which is contingent

III.5 Last comparison of modified RP and the modified BP

(P'') "*Possibly*, there exists a barber that shaves all and only those who do not shave themselves."

(C'') "*Necessarily*, there does not exist a barber that shaves all and only those who do not shave themselves."

(P'') "*Possibly*, there exists a set of all and only those sets which are not members of themselves."

(C'') "*Necessarily*, there does not exist a set of all and only those sets which are not members of themselves."

- existence of such and such barber is a plain empirical matter
- existence of such and such set is a matter in mathematical realm where consistency question is a *condicio sine qua non* (R would be an inconsistent multiplicity)

IV. Conclusion

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IV. Conclusion

- on the standard, Quinean definition of paradox, the *BP is not a paradox* at all, because it contains no problematic premise (naïve theory) which would be contradicted by the conclusion
- the crucial phrases of the BP and RP are only seemingly similar: one picks out an *empty set*, while the latter one picks out *no set*
- the appearance of *similarity* is based on a mistake (inclusion instead of membership)
- *paradoxicality* of the BP stems from a confusion of reflexive and irreflexive reading of the verb “shave” which have distinct existence consequences

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