Matematická logika

Důkazové kalkuly, Kalkul Hilbertova typu (12. přednáška)

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OP Vzdělávání pro konkurenceschopnost

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Formal systems, Proof calculi

A ***proof calculus*** (of a theory) is given by:

A. a ***language***
B. a set of ***axioms***
C. a set of ***deduction rules***

ad A. The definition of a ***language*** of the system consists of:

- an ***alphabet*** (a non-empty set of symbols), and
- a ***grammar*** (defines in an inductive way a set of well-formed formulas - WFF)

Hilbert-like calculus.
Language: restricted FOPL

Alphabet:

1. logical symbols:
    (countable set of) individual ***variables*** *x, y, z, …*
    connectives **¬, ⊃**
    quantifiers ∀
2. special symbols (of arity *n*)
    predicate symbols P^n, Q^n, R^n, …
    functional symbols f^n, g^n, h^n, …
    constants *a, b, c,* – functional symbols of arity 0
3. auxiliary symbols (,), [,], …

Grammar:

1. terms
    each ***constant*** and each ***variable*** is an *atomic term*
    if $t_1, \ldots, t_n$ are terms, f^n a functional symbol, then $\mathbf{f^n(t_1, \ldots, t_n)}$ is a *(functional) term*
2. atomic formulas
    if $t_1, \ldots, t_n$ are terms, P^n predicate symbol, then $\mathbf{P^n(t_1, \ldots, t_n)}$ is an *atomic (well-formed) formula*
3. composed formulas
    Let A, B be well-formed formulas. Then **¬A, (A⊃B)**, are *well-formed formulas*.
    Let A be a well-formed formula, *x* a variable. Then **∀*x*A** is a *well-formed formula*.
4. Nothing is a WFF unless it so follows from 1.-3.

Hilbert calculus

Ad B. The set of ***axioms*** is a chosen subset of the set of WFF.

The set of axioms has to be decidable: *axiom schemes*:

1. $A \supset (B \supset A)$
2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
3. $(\neg B \supset \neg A) \supset (A \supset B)$
4. $\forall x\, A(x) \supset A(x/t)$ Term t substitutable for *x* in A

▯. $(\forall x\, [A \supset B(x)]) \supset (A \supset \forall x\, B(x))$, *x* is not free in A

Hilbert calculus

Ad C. The ***deduction rules*** are of a form:

$$\mathbf{A_1,\ldots,A_m \mid\!- \; B_1,\ldots,B_m}$$

enable us to prove ***theorems*** (*provable formulas*) of the calculus. We say that each B_i is *derived* (inferred) from the set of assumptions $A_1,\ldots,A_m$.

Rule schemas:

MP: $A, A \supset B \mid\!- B$ (modus ponens)
G: $A \mid\!- \forall x\, A$ (generalization)

Hilbert calculus

Notes:

1. A, B are not formulas, but meta-symbols denoting any formula. Each axiom schema denotes an infinite class of formulas of a given form. If axioms were specified by concrete formulas, like

   1. $p \supset (q \supset p)$
   2. $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$
   3. $(\neg q \supset \neg p) \supset (p \supset q)$

   we would have to extend the set of rules with the *rule of substitution*:

   Substituting in a proved formula for each propositional logic symbol another formula, then the obtained formula is proved as well.

Hilbert calculus

2. The axiomatic system defined in this way works only with the symbols of connectives ¬, ⊃, and quantifier ∀. Other symbols of the other connectives and existential quantifier can be introduced as abbreviations *ex definicione*:

$$A \wedge B =_{df} \neg(A \supset \neg B)$$
$$A \vee B =_{df} (\neg A \supset B)$$
$$A \equiv B =_{df} ((A \supset B) \wedge (B \supset A))$$
$$\exists x A =_{df} \neg \forall x \neg A$$

The symbols ∧, ∨, ≡ and ∃ do not belong to the alphabet of the language of the calculus.

3. In Hilbert calculus we ***do not use the indirect proof***.

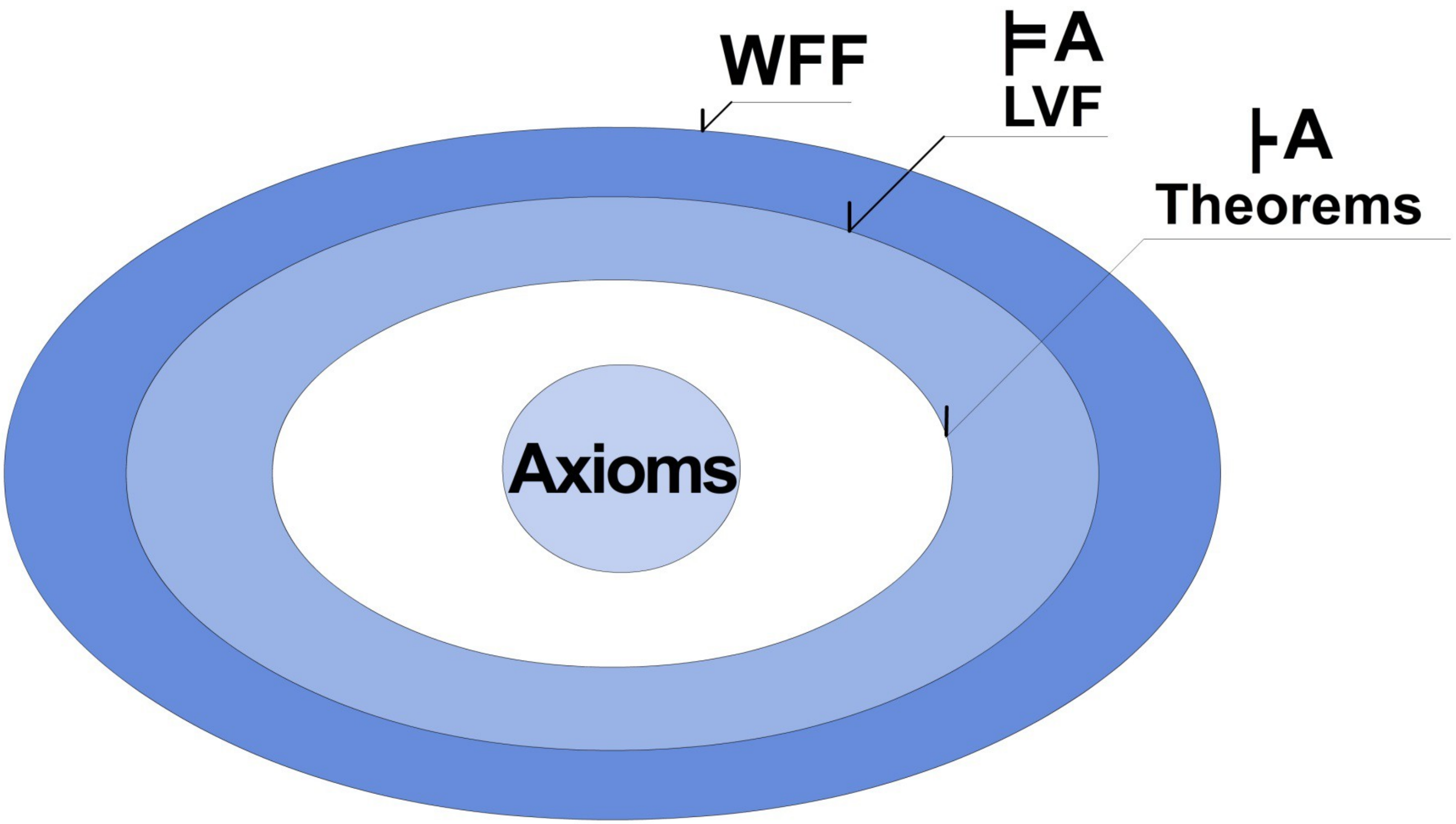
Hilbert calculus

4. Hilbert calculus defined in this way is ***sound (semantically consistent)***.

   a) **All the axioms are logically valid formulas**.
   b) **The modus ponens rule is truth-preserving**.

- The only problem – as you can easily see – is the ***generalisation rule***.
- This rule is obviously not truth preserving: formula **P(*x*) ⊃ ∀*x*P(*x*)** is not logically valid. However, this rule is ***tautology preserving***:
- **If the formula P(*x*) at the left-hand side is logically valid (or true in an interpretation), then ∀*x*A(*x*) is logically valid (or true in an interpretation) as well.**
- Since the axioms of the calculus are logically valid, the rule is correct.
- After all, this is a common way of proving in mathematics. To prove that something holds for all the triangles, we prove that for *any* triangle.

A sound calculus:
if |- A (provable) then |= A (True)



Proof in a calculus

- ***A proof of a formula A*** (from logical axioms of the given calculus) is a sequence of formulas (proof steps) $B_1, \ldots, B_n$ such that:
- $A = B_n$ (the proved formula A is the last step)
- each B_i $(i=1,\ldots,n)$ is either
  - an axiom or
  - B_i is derived from the previous B_j $(j=1,\ldots,i-1)$ using a deduction rule of the calculus.
- A formula A is ***provable*** by the calculus, denoted **|– A**, if there is a proof of A in the calculus. A provable formula is called a ***theorem.***

Hilbert calculus

- Note that any axiom is a theorem as well. Its proof is a trivial one step proof.

- To make the proof more comprehensive, you can use in the proof sequence also ***previously proved formulas (theorems)***.

- Therefore, we will first prove the rules of natural deduction, transforming thus Hilbert Calculus into the natural deduction system.

A Proof from Assumptions

A (direct) proof of a formula A from assumptions $A_1,\dots,A_m$ is a sequence of formulas (proof steps) $B_1,\dots B_n$ such that:

- $A = B_n$ (the proved formula A is the last step)
- each B_i ($i=1,\dots,n$) is either
  - an axiom, or
  - an assumption A_k ($1 \le k \le m$), or
  - B_i is derived from the previous B_j ($j=1,\dots,i-1$) using a rule of the calculus.

A formula ***A*** is **provable from $A_1, \dots, A_m$**, denoted $A_1, \dots, A_m \vdash A$, if there is a proof of A from $A_1,\dots,A_m$.

Examples of proofs (sl. 4)

Proof of a formula schema $A \supset A$:

1. $\mathbf{(A \supset ((A \supset A) \supset A)) \supset ((A \supset (A \supset A)) \supset (A \supset A))}$ axiom A2: B/A ⊃ A, C/A
2. $\mathbf{A \supset ((A \supset A) \supset A)}$ axiom A1: B/A ⊃ A
3. $\mathbf{(A \supset (A \supset A)) \supset (A \supset A)}$ MP:2,1
4. $\mathbf{A \supset (A \supset A)}$ axiom A1: B/A
5. $\mathbf{A \supset A}$ MP:4,3 Q.E.D.

Hence: $\vdash A \supset A$.

Examples of proofs

Proof of: **A ⊃ B, B ⊃ C** |- **A ⊃ C** (transitivity of implication TI):

1. **A ⊃ B** assumption
2. **B ⊃ C** assumption
3. **(A ⊃ (B ⊃ C)) ⊃ ((A ⊃ B) ⊃ (A ⊃ C))** axiom A2

4. **(B ⊃ C) ⊃ (A ⊃ (B ⊃ C))** axiom A1 A/(B ⊃ C), B/A
5. **A ⊃ (B ⊃ C)** MP:2,4
6. **(A ⊃ B) ⊃ (A ⊃ C)** MP:5,3
7. **A ⊃ C** MP:1,6 Q.E.D.

Hence: **A ⊃ B, B ⊃ C |- A ⊃ C** .

Examples of proofs

|- A(x/t) ⊃ ∃xA(x) (the ND rule – *existential generalisation*)

Proof:

1. **∀x ¬A(x) ⊃ ¬A(x/t)** axiom A4
2. **¬¬ ∀x ¬A(x) ⊃ ∀x ¬A(x)** theorem of type ¬¬C ⊃ C (see below)
3. **¬¬ ∀x ¬A(x) ⊃ ¬A(x/t)** C ⊃ D, D ⊃ E |– C ⊃ E: 2, 1 TI
4. **¬ ∀x ¬A(x) = ∃xA(x)** Intr. ∃ acc. (by definition)
5. **¬∃xA(x) ⊃ ¬A(x/t)** substitution: 4 into 3
6. **[¬∃xA(x) ⊃ ¬A(x/t)] ⊃ [A(x/t) ⊃ ∃xA(x)]** axiom A3
7. **A(x/t) ⊃ ∃xA(x)** MP: 5, 6 Q.E.D.

Examples of proofs

$\mathbf{A \supset B(\mathit{x}) \mid\!- \ A \supset \forall \mathit{x} B(\mathit{x})}$ (x is not free in A)

Proof:

1. $\mathbf{A \supset B(\mathit{x})}$ assumption
2. $\mathbf{\forall \mathit{x}[A \supset B(\mathit{x})]}$ Generalisation:1
3. $\mathbf{\forall \mathit{x}[A \supset B(\mathit{x})] \supset [A \supset \forall \mathit{x} B(\mathit{x})]}$ axiom A5
4. $\mathbf{A \supset \forall \mathit{x} B(\mathit{x})}$ MP: 2,3 Q.E.D.

The Theorem of Deduction

- Let A be a ***closed*** formula, B any formula. Then:

  $\mathbf{A_1, A_2, \ldots, A_k} \mid\!- \ \mathbf{A \supset B}$ *if and only if* $\mathbf{A_1, A_2, \ldots, A_k, A} \mid\!- \ \mathbf{B}$.

***Remark*:** The statement

a) ***if*** $\mid\!- \ \mathbf{A \supset B}$, ***then*** $\mathbf{A} \mid\!- \ \mathbf{B}$

is valid universally, not only for A being a closed formula (the proof is obvious – modus ponens).

On the other hand, the other statement

b) ***If*** $\mathbf{A} \mid\!- \ \mathbf{B}$, ***then*** $\mid\!- \ \mathbf{A \supset B}$

is ***not valid*** for an open formula A (with at least one free variable).

- *Example*: Let A = A(x), B = $\forall x$A(x).

  Then $\mathbf{A(x)} \mid\!- \ \mathbf{\forall x A(x)}$ is valid according to the generalisation rule.

  But the formula $\mathbf{A(x) \supset \forall x A(x)}$ is generally not logically valid, and therefore not provable in a sound calculus.

The Theorem of Deduction

- ***Proof*** (we will prove the Deduction Theorem only for the propositional logic)**:**

1. → Let A_1, A_2,...,A_k |– A ⊃ B.

Then there is a sequence $\mathbf{B_1, B_2,\ldots,B_n}$, which is the proof of **A ⊃ B** from assumptions $\mathbf{A_1,A_2,\ldots,A_k}$.

The proof of **B from** $\mathbf{A_1, A_2,\ldots,A_k}$**, A** is then the sequence of formulas $\mathbf{B_1, B_2,\ldots,B_n}$**, A, B,** where $\mathbf{B_n}$ **= A ⊃ B** and **B** is the result of applying modus ponens to formulas $\mathbf{B_n}$ and **A**.

The Theorem of Deduction

2. $\Leftarrow$ **Let $A_1, A_2, \ldots, A_k, A \vdash B$.**

Then there is a sequence of formulas **$C_1, C_2, \ldots, C_r \models B$**, which is the proof of **B from $A_1, A_2, \ldots, A_k$, A.** We will prove by induction that the formula **$A \supset C_i$** (for all i = 1, 2,...,r) is provable **from $A_1, A_2, \ldots, A_k$**. Then also **$A \supset C_r$** will be proved.

a) *Base of the induction*: If the length of the proof is = 1, then there are possibilities:

1. C_1 is an assumption A_i, or axiom, then:
2. $C_1 \supset (A \supset C_1)$ axiom A1
3. $A \supset C_1$ MP: 1,2

Or, In the third case $C_1 = A$, and we are to prove $A \supset A$ (see example 1).

b) *Induction step*: we prove that on the assumption of **$A \supset C_n$** being proved for ***n* = 1, 2, ..., i-1** the formula

$A \supset C_n$ can be proved also for ***n* = *i***.
For **C_i** there are four cases:

1. C_i is an assumption of A_i,
2. C_i is an axiom,
3. C_i is the formula A,
4. C_i is an immediate consequence of the formulas C_j and $C_k = (C_j \supset C_i)$, where j, k < i. In the first three cases the proof is analogical to a). In the last case the proof of the formula **$A \supset C_i$** is the sequence of formulas:

1. **$A \supset C_j$** induction assumption
2. **$A \supset (C_j \supset C_i)$** induction assumption
3. **$(A \supset (C_j \supset C_i)) \supset ((A \supset C_j) \supset (A \supset C_i))$** A2
4. **$(A \supset C_j) \supset (A \supset C_i)$** MP: 2,3
5. **$(A \supset C_i)$** MP: 1,4 Q.E.D

Semantics

- A semantically correct (sound) ***logical calculus*** serves for ***proving logically valid formulas*** (tautologies). In this case the
- ***axioms*** have to be ***logically valid formulas*** (true under all interpretations), and the
- ***deduction rules*** have to make it possible to prove logically valid formulas. For that reason the rules are ***either truth-preserving or tautology preserving***, i.e., $A_1,\dots,A_m \mid\!- \; B_1,\dots,B_m$ can be read as follows:
  - if all the formulas $A_1,\dots,A_m$ are logically valid formulas, then $B_1,\dots,B_m$ are logically valid formulas.

Theorem on Soundness (semantic consistence)

- Each provable formula in the Hilbert calculus is also logically valid formula: ***If* |- A, *then* |= A.**

Proof *(outline)*:

- Each formula of the form of an axiom schema of A1 – A5 is logically valid (i.e. true in every interpretation structure I for any valuation *v* of free variables).

- Obviously, MP (*modus ponens*) is a truth preserving rule.

- Generalisation rule: **A(*x*) |- ∀*x*A(*x*)** ?

Theorem on Soundness (semantic consistence)

- Generalisation rule **A(*x*) |– ∀*x*A(*x*)** is tautology preserving:

- Let us assume that **A(*x*)** is a proof step such that in the sequence preceding **A(*x*)** the generalisation rule has not been used as yet.

- Hence |= **A(*x*)** (since it has been obtained from logically valid formulas by using at most the truth preserving *modus ponens* rule).

- It means that in *any structure I the formula A(x) is true for any valuation e of x.* Which, by definition, means that **|= ∀*x*A(*x*)** (is logically valid as well).

Hilbert & natural deduction

- According to the Deduction Theorem each theorem of the implication form corresponds to a deduction rule(s), and vice versa.

  For example:

Theorem	Rule(s)
$\vdash A \supset ((A \supset B) \supset B)$	**$A, A \supset B \vdash B$** (MP rule)
$\vdash A \supset (B \supset A)$ ax. A1	**$A \vdash B \supset A$**; **$A, B \vdash A$**
$\vdash A \supset A$	**$A \vdash A$**
$\vdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$	**$A \supset B \vdash (B \supset C) \supset (A \supset C)$; $A \supset B, B \supset C \vdash A \supset C$ /rule TI/**

Example: a few simple theorems and the corresponding (natural deduction) rules:

1.	$\vdash A \supset (\neg A \supset B)$; $\vdash \neg A \supset (A \supset B)$	$A, \neg A \vdash B$	
2.	$\vdash A \supset A \vee B$; $\vdash B \supset A \vee B$	$A \vdash A \vee B$; $B \vdash A \vee B$	ID
3.	$\vdash \neg\neg A \supset A$	$\neg\neg A \vdash A$	EN
4.	$\vdash A \supset \neg\neg A$	$A \vdash \neg\neg A$	IN
5.	$\vdash (A \supset B) \supset (\neg B \supset \neg A)$	$A \supset B \vdash \neg B \supset \neg A$	TR
6.	$\vdash A \wedge B \supset A$; $\vdash A \wedge B \supset B$	$A \wedge B \vdash A, B$	EC
7.	$\vdash A \supset (B \supset A \wedge B)$; $\vdash B \supset (A \supset A \wedge B)$	$A, B \vdash A \wedge B$	IC
8.	$\vdash A \supset (B \supset C) \supset (A \wedge B \supset C)$	$A \supset (B \supset C) \vdash A \wedge B \supset C$	

Some proofs

Ad 1. **$\vdash$ A ⊃ (¬A ⊃ B)**; i.e.: **A, ¬A $\vdash$ B**.

Proof: (***from a contradiction $\vdash$ anything***)

1.	A	assumption
2.	¬A	assumption
3.	(¬B ⊃ ¬A) ⊃ (A ⊃ B)	A3
4.	¬A ⊃ (¬B ⊃ ¬A)	A1
5.	¬B ⊃ ¬A	MP: 2,4
6.	A ⊃ B	MP: 5,3
7.	B	MP: 1,6 Q.E.D.

Some proofs

Ad 2. **$\vdash A \supset A \vee B$**, i.e.: **$A \vdash A \vee B$**.
(the ***rule ID of the natural deduction***)

Using the definition abbreviation

$A \vee B =_{df} \neg A \supset B$,

we are to prove the theorem:

$\vdash A \supset (\neg A \supset B)$, i.e.

the rule **$A, \neg A \vdash B$**, which has been already proved.

Some proofs

Ad 3. |- ¬¬A ⊃ A; i.e.: **¬¬A |- A**.

Proof:

1. ¬¬A assumption
2. (¬A ⊃ ¬¬¬A) ⊃ (¬¬A ⊃ A) axiom A3
3. ¬¬A ⊃ (¬A ⊃ ¬¬¬A) theorem ad 1.
4. ¬A ⊃ ¬¬¬A MP: 1,3
5. ¬¬A ⊃ A MP: 4,2
6. A MP: 1,5 Q.E.D.

Some proofs

Ad 4. **$\vdash$ A $\supset$ ¬¬A;** i.e.: **A $\vdash$ ¬¬A**.

Proof:

1. A — assumption
2. (¬¬¬A ⊃ ¬A) ⊃ (A ⊃ ¬¬A) — axiom A3
3. ¬¬¬A ⊃ ¬A — theorem ad 3.
4. A ⊃ ¬¬A — MP: 3,2 Q.E.D.

Some proofs

Ad 5. **$\vdash$ (A ⊃ B) ⊃ (¬B ⊃ ¬A)**, i.e.: (**A ⊃ B) $\vdash$ (¬B ⊃ ¬A)**.

Proof:

1. A ⊃ B assumption
2. ¬¬A ⊃ A theorem ad 3.
3. ¬¬A ⊃ B TI: 2,1
4. B ⊃ ¬¬B theorem ad 4.
5. A ⊃ ¬¬B TI: 1,4
6. ¬¬A ⊃ ¬¬B TI: 2,5
7. (¬¬A ⊃ ¬¬B) ⊃ (¬B ⊃ ¬A) axiom A3
8. ¬B ⊃ ¬A MP: 6,7 Q.E.D.

Some proofs

Ad 6. **|– $(A \wedge B) \supset A$**, i.e.: **$A \wedge B$ |– A**. (The ***rule EC of the natural deduction***)

Using definition abbreviation **$A \wedge B =_{df} \neg(A \supset \neg B)$** we are to prove

|– $\neg(A \supset \neg B) \supset A$, i.e.: **$\neg(A \supset \neg B)$ |– A**.

Proof:

1. $\neg(A \supset \neg B)$ assumption
2. $(\neg A \supset (A \supset \neg B)) \supset (\neg(A \supset \neg B) \supset \neg\neg A)$ theorem ad 5.
3. $\neg A \supset (A \supset \neg B)$ theorem ad 1.
4. $\neg(A \supset \neg B) \supset \neg\neg A$ MP: 3,2
5. $\neg\neg A$ MP: 1,4
6. $\neg\neg A \supset A$ theorem ad 3.
7. A MP: 5,6 Q.E.D.

Some meta-rules

Let T is any finite set of formulas: T = $\{A_1, A_2, .., A_n\}$. Then

(a) *if* T, A |– B and |– A, *then* T |– B.
It is not necessary to state theorems in the assumptions.
(b) *if* A |– B, *then* T, A |– B. (Monotonicity of proving)
(c) *if* T |– A and T, A |– B, *then* T |– B.
(d) *if* T |– A and A |– B, *then* T |– B.
(e) *if* T |– A; T |– B; A, B |– C *then* T |– C.
(f) *if* T |– A and T |– B, *then* T |– A $\wedge$ B.
(Consequences can be composed in a conjunctive way.)
(g) T |– A ⊃ (B ⊃ C) *if and only if* T |– B ⊃ (A ⊃ C).
(The order of assumptions is not important.)
(h) T, A $\vee$ B |– C *if and only if* **both T, A |– C and T, B |– C**.
(Split the proof whenever there is a disjunction in the sequence – meta-rule of the natural deduction)
(i) *if* T, A |– B *and if* T, ¬A |– B, *then* T |– B.

Proofs of meta-rules (a sequence of rules)

Ad (h) $\Rightarrow$: Let T, A $\vee$ B |– C, we prove that: T, A |– C; T, B |– C.

Proof:

1. A |– A $\vee$ B the rule ID
2. T, A |– A $\vee$ B meta-rule (b): 1
3. T, A $\vee$ B |– C assumption
4. T, A |– C meta-rule (d): 2,3 Q.E.D.
5. T, B |– C analogically to 4. Q.E.D.

Proofs of meta-rules (a sequence of rules)

Ad (h) ⇐: Let T, A |– C; T, B |– C, we prove that T, A ∨ B |– C.

Proof:

1. **T, A |– C** assumption
2. **T |– A ⊃ C** deduction Theorem:1
3. **T |– ¬C ⊃ ¬A** meta-rule (d): 2,(the rule **TR** of natural deduction)
4. **T, ¬C |– ¬A** deduction Theorem: 3
5. **T, ¬C |– ¬B** analogical to 4.
6. **T, ¬C |– ¬A ∧ ¬B** meta-rule (f): 4,5
7. **¬A ∧ ¬B |– ¬(A ∨ B)** de Morgan rule (prove it!)
8. **T, ¬C |– ¬(A ∨ B)** meta-rule (d): 6,7
9. **T |– ¬C ⊃ ¬(A ∨ B)** deduction theorem: 8
10. **T |– A∨ B ⊃ C** meta-rule (d): 9. (the rule TR)
11. **T, A ∨ B |– C** deduction theorem: 10 Q.E.D.

Proofs of meta-rules (a sequence of rules)

Ad (i): Let T, A |– B; T, ¬A |– B, we prove T |– B.

Proof:

1. T, A |– B assumption
2. T, ¬A |– B assumption
3. T, A ∨ ¬A |– B meta-rule (h): 1,2
4. T |– B meta-rule (a): 3

A Complete Calculus: if |= A then |- A

- Each logically valid formula is provable in the calculus.

- The set of theorems = the set of logically valid formulas (the red sector of the previous slide is empty).

- ***Sound (semantic consistent) and complete calculus***:

  |= A iff |- A

  - Provability and logical validity coincide in FOPL (1^{st}-order predicate logic).

- ***Hilbert calculus is sound and complete.***

Properties of a calculus: deduction rules, consistency

- ***The set of deduction rules*** enables us to perform proofs mechanically, considering just the symbols, abstracting of their semantics. Proving in a calculus is a ***syntactic method***.

- A natural demand is a ***syntactic consistency*** of the calculus.

- A ***calculus is consistent*** iff there is a WFF ϕ such that ϕ is not provable (*in an inconsistent calculus everything is provable*).

- This definition is equivalent to the following one: a calculus is consistent iff a formula of the form $A \wedge \neg A$, or $\neg(A \supset A)$, is not provable.

- A calculus is syntactically consistent iff it is sound (semantically consistent).

Sound and Complete Calculus: |= A iff |- A

- ***Soundness*** *(an outline of the proof has been done)*

- *In 1928 Hilbert and Ackermann published a concise small book Grundzüge der theoretischen Logik, in which they arrived at exactly this point: they had defined axioms and derivation rules of predicate logic (slightly distinct from the above), and formulated the problem of completeness. They raised a question whether such a proof calculus is complete in the sense that each logical truth is provable within the calculus; in other words, whether the calculus proves exactly all the logically valid FOPL formulas.*

- ***Completeness Proof:***

- ***Stronger version: if T |= ϕ, then T |- ϕ. Kurt Gödel, 1930***

- A ***theory T is consistent*** iff there is a formula ϕ which is not provable in T: *not T* **|- ϕ.**

Strong Completeness of Hilbert Calculus:
if T |= ϕ, then T |- ϕ

- The proof of the Completeness theorem is based on the following ***Lemma***:

***Each consistent theory has a model*.**

- ***if T |= ϕ, then T |- ϕ*** iff
- **if *not* T |- ϕ, then *not* T |= ϕ** $\Rightarrow$
- **{T** $\cup$ $\neg\phi$**}** does not prove ϕ as well (¬ϕ does not contradict T) $\Rightarrow$
- **{T** $\cup$ $\neg\phi$**}** is consistent, it has a model M $\Rightarrow$
- M is a model of T in which ϕ is not true $\Rightarrow$
- ϕ is not entailed by T: **T |= ϕ**

Properties of a calculus: Hilbert calculus is not decidable

- There is another property of calculi. To illustrate it, let's raise a question: having a formula ϕ, does the calculus *decide* ϕ?

- In other words, ***is there an algorithm*** that would answer Yes or No, having ϕ as input and answering the question whether ϕ is logically valid or no**?** If there is such an algorithm, then the calculus is ***decidable*****.**

- If the calculus is complete, then it proves all the logically valid formulas, and the proofs can be described in an algorithmic way.

- However, in case the input formula ϕ is not logically valid, the algorithm does not have to answer (in a final number of steps).

- Indeed, there are *no decidable 1st order predicate logic calculi, i.e.,* ***the problem of logical validity is not decidable*** in the FOPL.

- (the consequence of Gödel Incompleteness Theorems)

Provable = logically true? Provable from … = logically entailed by …?

- The relation of ***provability* ($A_1,\ldots,A_n$ |- A)** and the relation of *logical entailment* ($A_1,\ldots,A_n$ |= A) are ***distinct relations*.**

- Similarly, the **set of theorems |- A** (of a calculus) is generally not identical to **the set of logically valid formulas |= A**.

- The former is *syntactic and defined within a calculus,* the latter *independent of a calculus, it is semantic.*

- In a *sound* calculus the set of theorems is a *subset* of the set of logically valid formulas.

- In a *sound and complete* calculus the set of theorems is *identical* with the set of logically valid formulas.

Hilbert Calculus