# Matematická logika

## Principy důkazových kalkulů

## Přirozená dedukce (11.přednáška)

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# Formal systems, Proof calculi

A **proof calculus** (of a theory) is given by:

1. a ***language***
2. a set of ***axioms***
3. a set of ***deduction rules***

ad 1. The definition of a **language** of the system consists of:

- an ***alphabet*** (a non-empty set of symbols), and
- a ***grammar*** (defines in an inductive way a set of well-formed formulas - WFF)

# Proof calculi: Example of a language: FOPL

**Alphabet**:

1. logical symbols:
   (countable set of) individual ***variables*** *x, y, z, …*
   ***connectives*** ¬, ∧, ∨, ⊃, ≡
   ***quantifiers*** ∀, ∃
2. special symbols (of arity *n*)
   ***predicate symbols*** $P^n$, $Q^n$, $R^n$, …
   ***functional symbols*** $f^n$, $g^n$, $h^n$, …
   constants *a, b, c,* – functional symbols of arity 0
3. auxiliary symbols (, ), [, ], …

**Grammar**:

1. terms
   each ***constant*** and each ***variable*** is an *atomic term*
   if $t_1, \dots, t_n$ are terms, $f^n$ a functional symbol, then $\boldsymbol{f^n(t_1, \dots, t_n)}$ is a *(functional) term*
2. atomic formulas
   if $t_1, \dots, t_n$ are terms, $P^n$ predicate symbol, then $\boldsymbol{P^n(t_1, \dots, t_n)}$ is an *atomic (well-formed) formula*
3. composed formulas
   Let A, B be well-formed formulas. Then ¬A, (A∨B), (A∧B), (A⊃B), (A≡B), are *well-formed formulas*.
   Let A be a well-formed formula, *x* a variable. Then ∀*x*A, ∃*x*A are *well-formed formulas*.
4. Nothing is a WFF unless it so follows from 1.-3.

# Proof calculi

*Ad 2. The set of* ***axioms*** is a chosen subset of the set of WFF.

- The axioms are considered to be basic (*logically true*) formulas that are not being proved.

- *Example*: $\{p \vee \neg p, p \supset p\}$.

*Ad 3.* The ***deduction rules*** are of a form: $\mathbf{A_1,\ldots,A_m \mid\!- \; B_1,\ldots,B_m}$

- *Enable us to prove* ***theorems*** (*provable formulas*) of the calculus. We say that each $B_i$ is *derived* (inferred) from the set of assumptions $A_1,\ldots,A_m$.

- *Examples*: $p \supset q, p \mid\!- \; q$ (modus ponens)

$p \supset q, \neg q \mid\!- \; \neg p$ (modus tollendo tollens)

$p \wedge q \mid\!- \; p, q$ (conjunction elimination)

# Proof calculi

- **A proof of a formula A** (from logical axioms of the given calculus) is a sequence of formulas (proof steps) $B_1,\ldots, B_n$ such that:

  - $A = B_n$ (the proved formula $A$ is the last step)
  - each $B_i$ (i = 1,…,$n$) is either
    - an axiom or
    - $B_i$ is derived from the previous $B_j$ (j=1,…,i-1) using a deduction rule of the calculus.

- A formula $A$ is **provable** by the calculus, denoted **$\vdash A$**, if there is a proof of $A$ in the calculus. Provable formulas are **theorems** (of the calculus).

# A Proof from Assumptions

- **A (direct) proof of a formula A from assumptions $A_1,\ldots,A_m$** is a sequence of formulas (proof steps) $B_1,\ldots,B_n$ such that:
  - $A = B_n$ (the proved formula $A$ is the last step)
  - each $B_i$ (i=1,…,$n$) is either
    - an axiom, or
    - an ***assumption*** $A_k$ ($1 \leq k \leq m$), or
    - $B_i$ is derived from the previous $B_j$ (j=1,…,i-1) using a rule of the calculus.
- A formula $A$ is is **provable from $A_1, \ldots, A_m$**, denoted **$A_1,\ldots,A_m \mid\!- \; A$**, if there is a proof of A from $A_1,\ldots,A_m$.

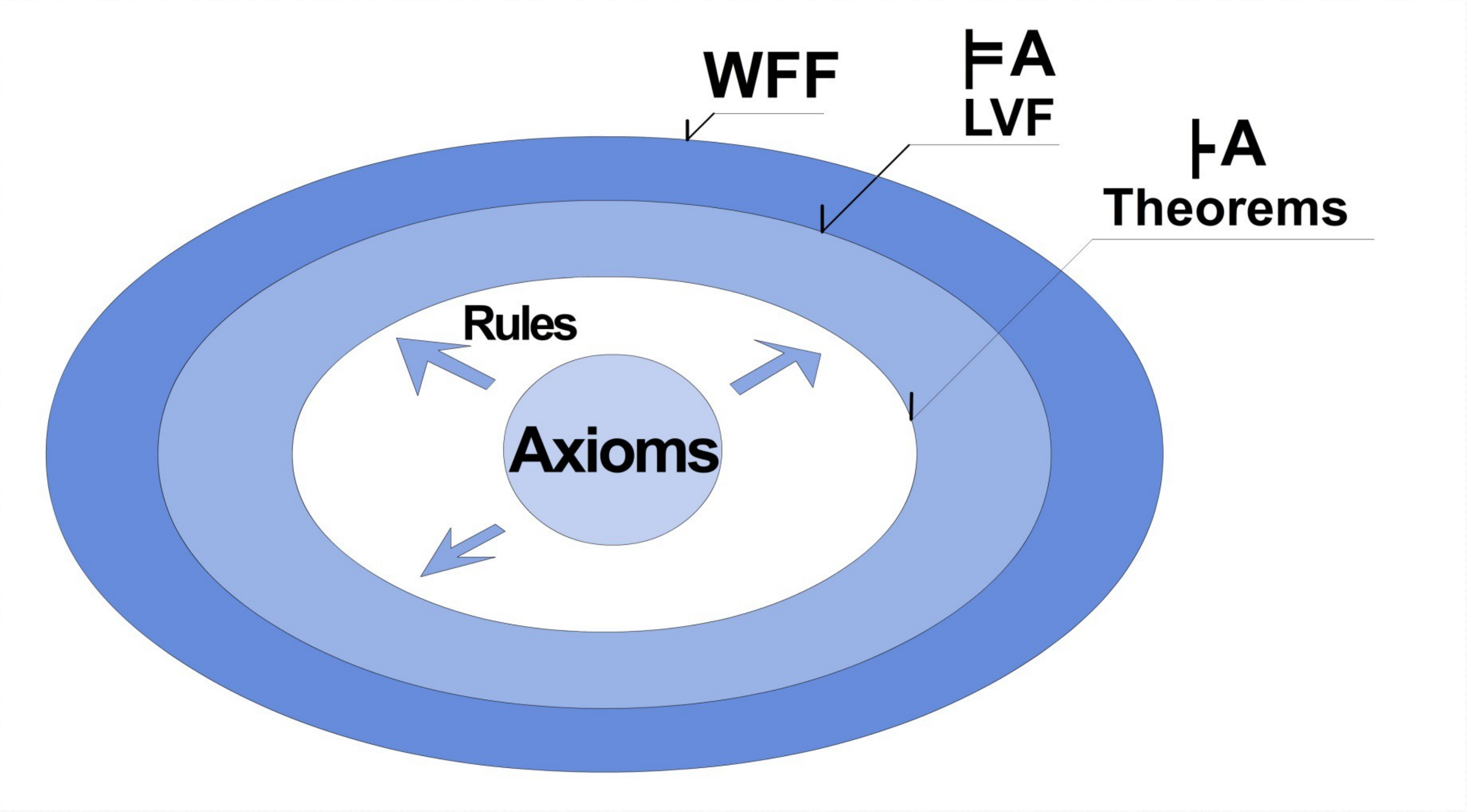
# A Proof from Assumptions

- **An indirect proof of a formula A from assumptions $A_1,\ldots,A_m$** is a sequence of formulas (proof steps) $B_1,\ldots,B_n$ such that:

  - each $B_i$ (i=1,...,$n$) is either

    - an axiom, or
    - an ***assumption*** $\boldsymbol{A_k}$ ($1 \leq k \leq m$), or
    - an ***assumption*** $\neg$***A of the indirect proof*** (formula *A* that is to be proved is negated)
    - $B_i$ is derived from the previous $B_j$ (j=1,...,i-1) using a rule of the calculus.
    - Some $B_k$ contradicts to $B_l$, i.e., $\boldsymbol{B_k = \neg B_l}$ ($k \in \{1,\ldots,n\}$, $l \in \{1,\ldots,n\}$)

# A sound calculus (sémanticky korektní):
# if |- $A$ (provable) then |= $A$ (logically true)



# A Complete Calculus: if |= *A* then |- *A*

- Each logically valid formula is provable in the calculus.

- The set of theorems = the set of logically valid formulas (the red sector of the previous slide is empty).

- ***Sound and complete calculus***:

  - **|= *A* iff |- *A***
  - Provability and logical validity coincide in FOPL (1st-order predicate logic)

- There are sound and complete calculi for the FOPL, e.g.: *Hilbert-like calculi, Gentzen calculi, natural deduction, resolution method*, …

# Semantics

- A semantically correct (sound) ***logical calculus*** serves for ***proving logically valid formulas*** (tautologies). In this case the
  - ***axioms*** have to be ***logically valid formulas*** (true under all interpretations), and the
  - ***deduction rules*** make it possible to prove logically valid formulas. For this reason the rules are ***either truth-preserving in general or preserving truth in an interpretation***, i.e., $A_1, \ldots, A_m \mid\!- \; B_1, \ldots, B_m$ can be read as follows:
  - if all the formulas $A_1, \ldots, A_m$ are logically valid formulas, then $B_1, \ldots, B_m$ are logically valid formulas.

# The Theorem of Deduction

- In a sound proof calculus the following Theorem of Deduction should be valid:

  ***Theorem of deduction*.** A formula $\phi$ is provable from assumptions $A_1,\ldots,A_m$, iff the formula $A_m \supset \phi$ is provable from $A_1,\ldots,A_{m-1}$.

  In symbols:

  $$A_1,\ldots,A_m \mid\!- \ \phi \text{ iff } A_1,\ldots,A_{m-1} \mid\!- \ (A_m \supset \phi).$$

- In a sound calculus meeting the Deduction Theorem the following implication holds:

  $$\text{If } \boldsymbol{A_1,\ldots,A_m \mid\!-} \ \phi \text{ then } \boldsymbol{A_1,\ldots,A_m \models} \ \phi.$$

- ***If the calculus is sound and complete, then provability coincides with logical entailment:***

  $\boldsymbol{A_1,\ldots,A_m \mid\!-} \ \phi$ iff $\boldsymbol{A_1,\ldots,A_m \models} \ \phi$.

# The Theorem of Deduction

**If the calculus is sound and complete, then provability coincides with logical entailment:**
**$A_1,\dots,A_m \vdash \phi$ iff $A_1,\dots,A_m \models \phi$.**

***Proof.*** If the Theorem of Deduction holds, then

$A_1,\dots,A_m \vdash \phi$ iff $\vdash (A_1 \supset (A_2 \supset \dots(A_m \supset \phi)\dots))$.
$\vdash (A_1 \supset (A_2 \supset \dots(A_m \supset \phi)\dots))$ iff $\vdash (A_1 \wedge \dots \wedge A_m) \supset \phi$.

- If the calculus is sound and complete, then

  $\vdash (A_1 \wedge \dots \wedge A_m) \supset \phi$ iff $\models (A_1 \wedge \dots \wedge A_m) \supset \phi$.
  $\models (A_1 \wedge \dots \wedge A_m) \supset \phi$ iff $A_1,\dots,A_m \models \phi$.

- The first equivalence is obtained by applying the Deduction Theorem *m*-times, the second is valid due to the soundness and completeness, the third one is the semantic equivalence.

# Properties of a calculus: axioms

- ***The set of axioms*** of a calculus is non-empty and ***decidable*** in the set of WFFs (otherwise the calculus would not be reasonable, for we couldn't perform proofs if we did not know which formulas are axioms).

- It means that there is an ***algorithm*** that for any WFF $\phi$ given as its input answers in a finite number of steps an output Yes or NO on the question whether $\phi$ is an axiom or not.

- A finite set is trivially decidable. The set of axioms can be infinite. In such a case we define the set either by an algorithm of creating axioms or by a finite set of ***axiom schemata.***

- The set of axioms should be ***minimal,*** i.e., each axiom is independent of the other axioms (not provable from them).

# Properties of a calculus: deduction rules, consistency

- ***The set of deduction rules*** enables us to perform proofs mechanically, considering just the symbols, abstracting of their semantics. Proving in a calculus is a ***syntactic method***.

- A natural demand is a ***syntactic consistency*** of the calculus.

- A ***calculus is consistent*** iff there is a WFF $\phi$ such that $\phi$ is not provable (*in an inconsistent calculus everything is provable*).

- This definition is equivalent to the following one: a calculus is consistent iff a formula of the form $A \wedge \neg A$, or $\neg(A \supset A)$, is not provable.

- ***A calculus is syntactically consistent iff it is sound (semantically consistent).***

# Properties of a calculus: (un)decidability

- There is another desirable property of calculi. To illustrate it, let’s raise a question: having a formula $\phi$, does the calculus *decide* $\phi$?
- In other words, ***is there an algorithm*** that would answer in a finite number of steps *Yes* or *No*, having $\phi$ as input and answering the question whether $\phi$ is logically valid or no**?** If there is such an algorithm, then the calculus is ***decidable.***
- If the calculus is complete, then it proves all the logically valid formulas, and the proofs can be described in an algorithmic way.
- However, in case the input formula $\phi$ is not logically valid, the algorithm does not have to answer (in a finite number of steps).
- Indeed, **there is *no decidable 1st order predicate logic calculus***, *i.e.*, ***the problem of logical validity is not decidable*** in the FOPL.
- (the consequence of Gödel Incompleteness Theorems)

# Provable = logically true?
# Provable from … = logically entailed by …?

- The relation of ***provability* ($A_1,\dots,A_n$ |- $A$)** and the relation of *logical entailment* ($A_1,\dots,A_n$ |= $A$) are ***distinct relations*.**
- Similarly, the **set of theorems |- $A$** (of a calculus) is generally not identical to **the set of logically valid formulas |= $A$**.
- The former is *syntactic and defined within a calculus,* the latter is *independent of a calculus, it is semantic.*
- In a *sound* calculus the set of theorems is a *subset* of the set of logically valid formulas.
- In a *sound and complete* calculus the set of theorems is *identical* with the set of formulas.

# „pre-Hilbert“ formalists

- „Mathematics is a game with symbols“
- A simple system S:
  Constants: ♣,♥
  Predicates: ◆♠
  Axioms of S: (1) ∀*x* (◆*x* ⊃ ♠*x*)
  (2) ∃*x* ♠*x* ⊃ ◆♣
  (3) ♠♥
  Inference rules: MP (modus ponens), E∀ (general quantifier elimination), I∃
  (existential quantifier insertion)

**Theorem:** ♠ ♣
**Proof**: ♠♥ (axiom 3)
∃*x* ♠*x* (I∃)
◆♣ (axiom 2 and MP)
◆♣ ⊃ ♠♣ (axiom 1 and E∀)
♠♣ (MP)

**It is impossible to develop mathematics in such a purely formalist way. Instead: use only finitist methods (Gödel: impossible as well)**

# Historical background

- The reason why proof calculi have been developed can be traced back to the end of 19th century. At that time *formalization methods* had been developed and various *paradoxes* arose. All those paradoxes arose from the assumption on the existence of ***actual infinities***.

- To avoid paradoxes, ***David Hilbert*** (a significant German mathematician) proclaimed the ***program of formalisation of mathematics.*** The idea was simple: to avoid paradoxes we will use only ***finitist methods***:

- First:
  - start with a decidable set of certainly (logically) true formulas,
  - use truth-preserving rules of deduction, and
  - infer all the logical truths.

- Second,
  - begin with some sentences true in an area of interest (interpretation),
  - use truth-preserving rules of deduction, and
  - infer all the truths of this area.

- In particular, he intended to ***axiomatise*** in this way ***mathematics***, in order to ***avoid paradoxes***.

# Historical background

- Hilbert supposed that these goals can be met.

- *Kurt* ***Gödel*** (the greatest logician of the 20th century) proved *the completeness of the 1st order predicate calculus*, which was expected. He even proved the strong completeness:

  **if *SA* |= *T* then *SA* |– *T*** (*SA* – a set of assumptions).

- But Hilbert wanted more: he supposed that ***all the truths of mathematics*** can be proved in this mechanic finite way. That is, that a ***theory of arithmetic*** (e.g. Peano) is ***complete*** in the following sense:
  - each formula is in the theory ***decidable***, i.e., the theory proves either the formula or its negation, which means that all the formulas true in *the intended interpretation* over the set of natural numbers are provable in the theory:

- Gödel's ***theorems on incompleteness*** give a surprising result: ***there are true but not provable sentences of natural numbers arithmetic***. Hence Hilbert program is not fully realisable.

# Natural Deduction Calculus

- ***Axioms***: **A $\vee$ ¬A, A ⊃ A**

- ***Deduction Rules*:**
  - ***conjunction***: **A, B |– A $\wedge$ B** (IC)

**A $\wedge$ B |– A, B** (EC)
  - ***disjunction***: **A |– A $\vee$ B or B |– A $\vee$ B** (ID)

**A $\vee$ B,¬A |– B or A $\vee$ B,¬B |– A** (ED)
  - ***Implication***: **B |– A ⊃ B** (II)

**A ⊃ B, A |– B** (EI, *modus ponens* MP)
  - ***equivalence***: **A ⊃ B, B ⊃ A |– A ≡ B** (IE)

**A ≡ B |– A ⊃ B, B ⊃ A** (EE)

# Natural Deduction Calculus

*Deduction rules for quantifiers*

- ***General quantifier***: **A(*x*) |– ∀*x*A(*x*) I∀**

  *The rule can be used only if formula A(x) is not derived from any assumption that would contain variable x as free.*

  **∀*x*A(*x*) |– A(*x*/*t*) E∀**

  *Formula A(x/t) is a result of correctly substituting the term t for the variable x (no collision of variables).*

- ***Existential quantifier*** **A(*x*/*t*) |– ∃*x*A(*x*) I∃**

  **∃*x*A(*x*) |– A(*x*/*c*) E∃**

  *where c is a constant **not used** in the language as yet. If the rule is used for distinct formulas, then a different constant has to be used. A more general form of the rule is:*

$\forall y_1 \ldots \forall y_n\, \exists x\, A(x, y_1,\ldots,y_n) \mid\!- \; \forall y_1 \ldots \forall y_n\, A(x / f(y_1,\ldots,y_n), y_1,\ldots,y_n)$ **General E∃**

# Natural Deduction (notes)

1. *In the natural deduction calculus an indirect proof is often used.*

2. *Existential quantifier elimination has to be done in accordance with the rules of Skolemisation in the general resolution method.*

3. *Rules derivable from the above:*

- **$A(x) \supset B \vdash \forall x A(x) \supset B$, $x$ is not free in B**
- **$A \supset B(x) \vdash A \supset \forall x B(x)$, $x$ is not free in A**
- **$A(x) \supset B \vdash \exists x A(x) \supset B$, $x$ is not free in B**
- **$A \supset B(x) \vdash A \supset \exists x B(x)$**
- **$A \supset \forall x B(x) \vdash A \supset B(x)$**
- **$\exists x A(x) \supset B \vdash A(x) \supset B$**

# Natural Deduction

*Another useful rules and theorems of propositional logic (try to prove them):*

***Introduction of negation***: **A |- ¬¬A IN**

***Elimination of negation***: **¬¬A |- A EN**

***Negation of disjunction***: **¬(A ˅ B) |- ¬A ˄ ¬B ND**

***Negation of conjunction***: **¬(A ˄ B) |- ¬A ˅ ¬B NK**

***Negation of implication***: **¬(A ⊃ B) |- A ˄ ¬B NI**

***Tranzitivity of implication***: **A ⊃ B, B ⊃ C |- A ⊃ C TI**

***Transpozition***: **A ⊃ B |- ¬B ⊃ ¬A TR**

***Modus tollens***: **A ⊃ B, ¬B |- ¬A MT**

# Natural Deduction: Examples

***Theorem 1*:**

**$A \supset B, \neg B \vdash \neg A$ Modus Tollens**

***Proof*:**

1. $A \supset B$ assumption
2. $\neg B$ assumption
3. $A$ assumption of the indirect proof
4. $B$ MP: 1, 3 contradicts to 2., hence

□. $\neg A$ Q.E.D

# Natural Deduction: Examples

***Theorem 2*:**

$$C \supset D \mid\!- \neg C \vee D$$

***Proof:***

1. $C \supset D$ assumption
2. $\neg(\neg C \vee D)$ assumption of indirect proof
3. $\neg(\neg C \vee D) \supset (C \wedge \neg D)$ de Morgan (see the next example)
4. $C \wedge \neg D$ MP 2,3
5. C EC 4
6. $\neg D$ EC 4
7. D MP 1, 5 contradicts to 6, hence
8. $\neg C \vee D$ (assumption of indirect proof is not true)Q.E.D.

# Proof of an implicative formula

- If a formula ***F*** is of an implicative form:

$$\mathbf{A_1 \supset \{A_2 \supset [A_3 \supset \ldots \supset (A_n \supset B) \ldots]\} \; (*)}$$

- then according to the Theorem of Deduction the formula ***F*** can be proved in such a way that the formula ***B*** is proved from the assumptions $\mathbf{A_1, A_2, A_3, \ldots, A_{\mathit{n}}}$.

# The technique of branch proof from hypotheses

- Let the proof sequence contain a disjunction:

$$\boldsymbol{D_1 \vee D_2 \vee \ldots \vee D_k}$$

- We introduce hypotheses $D_i$ ($1 \leq i \leq k$). If a formula $F$ can be proved from each of the hypotheses $D_i$, then $F$ is proved.

- *Proof (of the validity of branch proof)*:

a) *Theorem 4*: $\mathbf{[(p \supset r) \wedge (q \supset r)] \supset [(p \vee q) \supset r]}$
b) The rule II (implication introduction): $\mathbf{B \mid\!- A \supset B}$

# The technique of branch proof from hypotheses

*Theorem 4*:

**$[(p \supset r) \wedge (q \supset r)] \supset [(p \vee q) \supset r]$**

| | | |
|---|---|---|
| 1. | $[(p \supset r) \wedge (q \supset r)]$ | assumption |
| 2. | $(p \supset r)$ | EK: 1 |
| 3. | $(q \supset r)$ | EK: 1 |
| 4. | $p \vee q$ | assumption |
| 5. | $(p \supset r) \supset (\neg p \vee r)$ | *Theorem 2* |
| 6. | $\neg p \vee r$ | MP: 2.5. |
| 7. | $\neg r$ | assumption of the indirect proof |
| 8. | $\neg p$ | ED: 6.7. |
| 9. | $q$ | ED: 4.8. |
| 10. | $r$ | MP: 3.9. – contra 7., hence |
| 11. | $r$ | Q.E.D |

# The technique of branch proof from hypotheses

*Theorem 3*:

**$(\neg A \wedge \neg B) \supset \neg(A \vee B)$ de Morgan law**

*Proof:*

| | | | |
|---|---|---|---|
| 1. | $(\neg A \wedge \neg B)$ | | assumption |
| 2. | $A \vee B$ | | assumption of the indirect proof |
| 3. | $\neg A$ | | EC 1. |
| 4. | $\neg B$ | | EC 1. |
| | 5.1. | $A$ | **hypothesis**: contradicts to 3 |
| | 5.2. | $B$ | **hypothesis**: contradicts to 4. |
| 5. | $A \supset \neg(A \vee B)$ | | II |
| 6. | $B \supset \neg(A \vee B)$ | | II |
| 7. | $[A \supset \neg(A \vee B)] \wedge [B \supset \neg(A \vee B)]$ | | IC 5,6 |
| 8. | $(A \vee B) \supset \neg(A \vee B)$ | | *Theorem* 4 |
| 9. | $\neg(A \vee B)$ | | MP 2, 8: Q.E.D. |

# Natural Deduction: examples

*Theorem 5*:

**$A \supset C, B \supset C \vdash (A \vee B) \supset C$**

*Proof:*

1. $A \supset C$ assumption
2. $\neg A \vee C$ *Theorem 2*
3. $B \supset C$ assumption
4. $\neg B \vee C$ *Theorem 2*
5. $A \vee B$ assumption
6. $\neg C$ assumption of indirect proof
7. $\neg B$ ED 4, 6
8. $\neg A$ ED 2, 6
9. $\neg A \wedge \neg B$ IC 7, 8
10. $(\neg A \wedge \neg B) \supset \neg(A \vee B)$ *Theorem* 3 (de Morgan)
11. $\neg(A \vee B)$ MP 9, 10 contradicts to 5., hence
12. $C$ (assumption of indirect proof is not true) Q.E.D.

# Natural Deduction: examples

*Some proofs of FOPL theorems*

**1) |- $\forall x\, [A(x) \supset B(x)] \supset [\forall x A(x) \supset \forall x B(x)]$**

*Proof:*

| | |
|---|---|
| 1. $\forall x\, [A(x) \supset B(x)]$ | assumption |
| 2. $\forall x\, A(x)$ | assumption |
| 3. $A(x) \supset B(x)$ | E$\forall$:1 |
| 4. $A(x)$ | E$\forall$:2 |
| 5. $B(x)$ | MP:3,4 |
| 6. $\forall x B(x)$ | I$\forall$:5 |

Q.E.D.

# Natural Deduction: examples

- According to the Deduction Theorem we prove theorems in the form of implication by means of the proof of consequent from antecedent:

- $\forall x\, [A(x) \supset B(x)] \mid\!\!-\; [\forall x A(x) \supset \forall x B(x)]$ iff

- $\forall x\, [A(x) \supset B(x)],\ \forall x A(x) \mid\!\!-\; \forall x B(x)$

# Natural Deduction: examples

**2**)**|-** $\neg\forall x\, A(x) \equiv \exists x\, \neg A(x)$ (De Morgan rule)

*Proof:*

$\Rightarrow$: 1. $\neg\forall x\, A(x)$ assumption
2. $\neg\exists x\, \neg A(x)$ assumption of indirect proof
3.1. $\neg A(x)$ hypothesis
3.2. $\exists x\, \neg A(x)$ I$\exists$: 3.1
4. $\neg A(x) \supset \exists x\, \neg A(x)$ II: 3.1, 3.2
5. $A(x)$ MT: 4,2
6. $\forall x\, A(x)$ Z$\forall$:5 contradicts to:1 Q.E.D.

$\Leftarrow$: 1. $\exists x\, \neg A(x)$ assumption
2. $\forall x\, A(x)$ assumption of indirect proof
3. $\neg A(c)$ E$\exists$:1
4. $A(c)$ E$\forall$:2
contradicts to:3 Q.E.D.

# Natural Deduction: examples

- *Note*: In the proof sequence we can introduce a *hypothetical assumption H* (in this case 3.1.) and derive *conclusion C from* this hypothetical assumption *H* (in this case 3.2.). As a *regular proof step* we then must introduce implication **H ⊃ C** (step 4.).

- According to the Theorem of Deduction this theorem corresponds to two rules of deduction:

$$\neg \forall \boldsymbol{x}\, \mathbf{A}(\boldsymbol{x}) \;|\!\!-\; \exists \boldsymbol{x}\, \neg \mathbf{A}(\boldsymbol{x})$$

$$\exists \boldsymbol{x}\, \neg \mathbf{A}(\boldsymbol{x}) \;|\!\!-\; \neg \forall \boldsymbol{x}\, \mathbf{A}(\boldsymbol{x})$$

# Natural Deduction: examples

**3) |– ¬∃*x* A(*x*) ≡ ∀*x* ¬A(*x*)** (De Morgan rule)

*Proof:*

⇒: 1. ¬∃*x* A(*x*) assumption
2.1. A(*x*) hypothesis
2.2. ∃*x* A(*x*) Z∃:2.1
3. A(*x*) ⊃ ∃*x* A(*x*) ZI: 2.1, 2.2
4. ¬A(*x*) MT: 3,1
5. ∀*x* ¬A(*x*) Z∀:4 Q.E.D.

⇐: 1. ∀*x* ¬A(*x*) assumption
2. ∃*x* A(*x*) assumption of indirect proof
3. A(c) E∃: 2
4. ¬A(c) E∀: 1 contradictss to: 3 Q.E.D.

- According to the Theorem of Deduction this theorem (3) corresponds to two rules of deduction:

  **¬∃*x* A(*x*) |– ∀*x* ¬A(*x*), ∀*x* ¬A(*x*) |– ¬∃*x* A(*x*)**

# Existential quantifier elimination

*Note:* In the second part of the proofs *ad* (2) and (3) the rule of existential quantifier elimination (E∃) has been used.

This rule is not truth preserving: the formula **∃x A(x) ⊃ A(c)** ***is not logically valid*** (cf. Skolem rule in the resolution method: the rule is satifiability preserving).

There are two ways of its using correctly:

- In an indirect proof (satisfiability!)
- As a an intermediate step that is followed by Introducing ∃ again

- The proofs *ad* (2) and (3) are examples of the former (indirect proofs). The following proof is an example of the latter:

# Natural Deduction

**4)** **|- $\forall x$ [A($x$) ⊃ B($x$)] ⊃ [ $\exists x$ A($x$) ⊃ $\exists x$ B($x$)]**

*Proof*:

1. $\forall x$ [A($x$) ⊃ B($x$)] assumption
2. $\exists x$A($x$) assumption
3. A(a) E$\exists$: 2
4. A(a) ⊃ B(a) E$\forall$: 1
5. B(a) MP: 3,4
6. $\exists x$B($x$) I$\exists$: 5

Q.E.D.

*Note*: this is another example of a correct using the rule E$\exists$.

# Natural Deduction

**5) |– $\forall x$ [A $\vee$ B($x$)] ≡ A $\vee$ $\forall x$B($x$)**, where A does not contain variable $x$ free

*Proof*:

⇒: 1. $\forall x$ [A $\vee$ B($x$)] assumption

2. A $\vee$ B($x$) E$\forall$: 1

3. A $\vee$ $\neg$A axiom

3.1. A 1. hypothesis

3.2. A $\vee$ $\forall x$B($x$) ZD: 3.1

4.1. $\neg$A 2. hypothesis

4.2. B($x$) ED: 2, 4.1

4.3. $\forall x$B($x$) I$\forall$: 4.2

4.4. A $\vee$ $\forall x$B($x$) ID: 4.3.

5. [A ⊃ (A $\vee$ $\forall x$B($x$))] $\wedge$ [$\neg$A ⊃ (A $\vee$ $\forall x$B($x$))] II + IC

6. (A $\vee$ $\neg$A) ⊃ (A $\vee$ $\forall x$B($x$)) theorem + MP 5

7. A $\vee$ $\forall x$B($x$) MP 6, 3

Q.E.D.

# Natural Deduction

**5)|– $\forall$*x* [A $\vee$ B(*x*)] ≡ A $\vee$ $\forall$*x*B(*x*)**, where A does not contain variable *x* free

*Proof*:

⇐: 1. A $\vee$ $\forall x$B($x$) Assumption, disjunction of hypotheses
2.1. A 1. hypothesis
2.2. A $\vee$ B($x$) ID: 2.1
2.3. $\forall x$ [A $\vee$ B($x$)] I$\forall$: 2.2
3. A $\supset$ $\forall x$ [A $\vee$ B($x$)]
4.1. $\forall$xB(x) 2. hypothesis
4.2. B($x$) E$\forall$: 3.1
4.3. A $\vee$ B($x$) ID: 3.2
4.4. $\forall x$ [A $\vee$ B($x$)] I$\forall$: 3.3
5. $\forall$xB(x) $\supset$ $\forall x$ [A $\vee$ B($x$)] II 4.1., 4.4.
6. [A $\vee$ $\forall$xB(x)] $\supset$ $\forall x$ [A $\vee$ B($x$)] Theorem, IC, MP – 3, 5
7. $\forall x$ [A $\vee$ B($x$)] MP 1, 6 Q.E.D.

# Natural Deduction

**6)** **|- (A($x$) ⊃ B) ⊃ (∀$x$A($x$) ⊃ B)**

*Proof*:

| | | |
|---|---|---|
| 1. | A($x$) ⊃ B | assumption |
| 2. | ∀$x$A($x$) | assumption |
| 3. | A($x$) | E∀: 2 |
| 5. | B | MP: 1,2 |
| | | Q.E.D. |

This theorem corresponds to the rule:

**A(x) ⊃ B |- ∀$x$A($x$) ⊃ B**