INVESTMENTS IN EDUCATION DEVELOPMENT

1 Tableaux in propositional logic

Exercise 1.1: Draw a finished tableau with the root

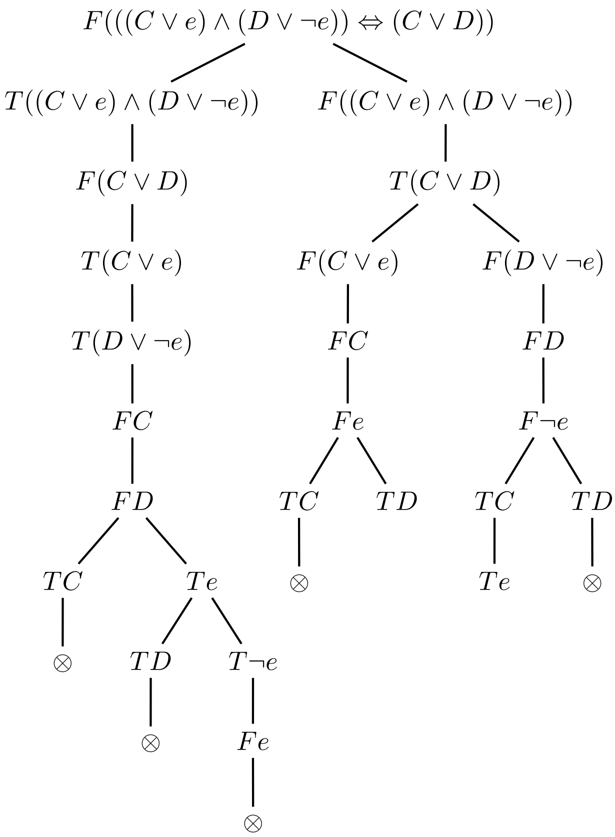
$$F(((C \vee e) \wedge (D \vee \neg e)) \Leftrightarrow (C \vee D))$$

where C, D, e are propositional letters. Discuss the changes in the tableau when implication $\Rightarrow$ is used instead of equivalence $\Leftrightarrow$ in the root node.

Solution 1.1: A finished tableau is a tree where every path is *finished.* A path is finished, if it is *contradictory* or if every node on the path is *reduced.* A path is *contradictory* (marked $\otimes$) if it contains nodes $T\varphi$ and $F\varphi$ for some proposition φ. A node E on a path P is *reduced* if there exists a path in the atomic tableau with the root E whose all nodes occur on P. If a tableau contains a node E that is not reduced on a non-contradictory path P, we adjoin the atomic tableau with the root E to the end of P.[1] By repeating this procedure we finally get a finished tableau.

When building *complete systematic tableau*, an unreduced node E nearest to the root is always selected (if there are more such nodes, the leftmost one is selected). We then adjoin the atomic tableau with the root E to the end of every noncontradictory path that goes through E and on which E is unreduced.

[1] Convention: the root E of the newly adjoined tableau is omitted – it already occurs on P.



The tableau in the picture is not systematic: in its right part we reduced the nodes $F(C \vee e)$ and $F(D \vee \neg e)$ before the node $T(C \vee D)$ that is nearer to the root.

The tableau contains paths that are not contradictory. Such paths contain interpretations which satisfy the root of the tableau (including the sign F). For example, the leftmost noncontradictory path in the tableau contains nodes TD, Fe, FC. It implies that for the interpretation $I(D) = 1, I(e) = 0, I(C) = 0$ the formula $((C \vee e) \wedge (D \vee \neg e)) \Leftrightarrow (C \vee D)$ is false and so the root is satisfied (because it requires that the formula is false using the sign F).

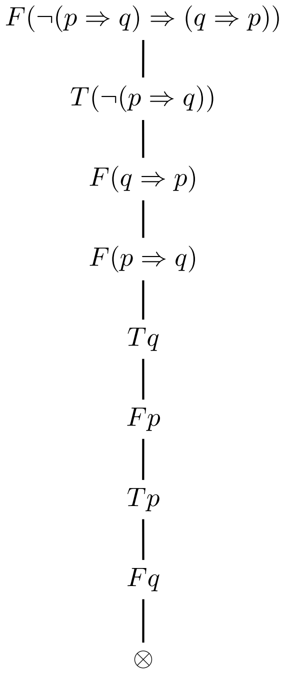
Notice that all the paths in the left part of the tableau are contradictory. If implication $\Rightarrow$ was used instead of equivalence $\Leftrightarrow$ in the root we would get a contradictory tableau. The tableau would be a proof of the implication that describes the resolution rule.

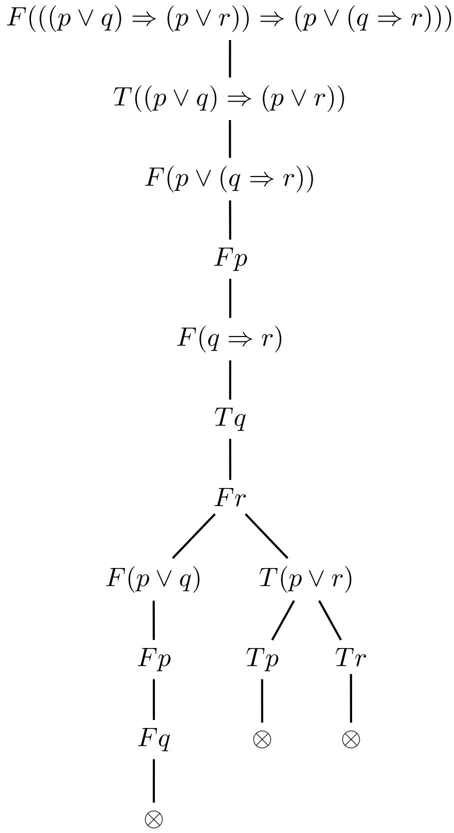
Exercise 1.2: Prove that the following formulas are tautologies using tableau method:

a) $\neg(p \Rightarrow q) \;\Rightarrow\; (q \Rightarrow p)$

b) $((p \vee q) \Rightarrow (p \vee r)) \;\Rightarrow\; (p \vee (q \Rightarrow r))$

Solution 1.2: The following finished contradictory tableaux can be built for the folmulas:

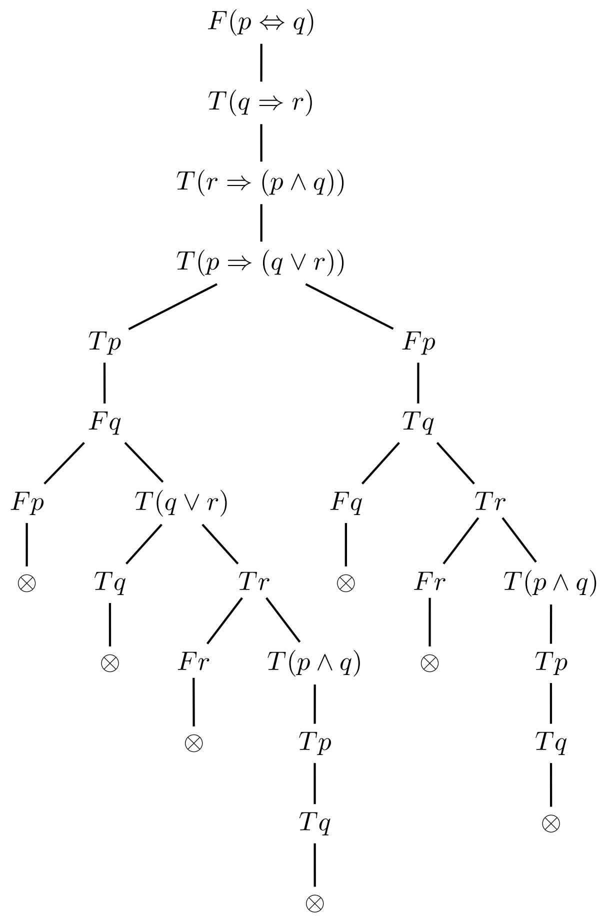




Exercise 1.3: Prove the following logical consequence:

$$\{q \Rightarrow r, r \Rightarrow (p \wedge q), p \Rightarrow (q \vee r)\} \models (p \Leftrightarrow q).$$

Solution 1.3: The conclusion is signed F and the premises are signed T. The signed conclusion is the root of the tableau and all of the signed premises are adjoined (to the same path). Then the nodes are reduced and finally a finished contradictory tableau is built. It proves that the logical consequence holds.


$F(p \Leftrightarrow q)$
$T(q \Rightarrow r)$
$T(r \Rightarrow (p \wedge q))$
$T(p \Rightarrow (q \vee r))$
Tp
Fp
Fq
Tq
Fp
$T(q \vee r)$
Fq
Tr
$\otimes$
Tq
Tr
$\otimes$
Fr
$T(p \wedge q)$
$\otimes$
Fr
$T(p \wedge q)$
$\otimes$
Tp
$\otimes$
Tp
Tq
Tq
$\otimes$
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