1 Tableaux in predicate logic

Exercise 1.1: Prove that the following formulas are tautologies. Use tableau method.

a) $\Phi_1 \equiv \forall x \varphi(x) \Rightarrow \neg(\exists x \neg \varphi(x))$

b) $\Phi_2 \equiv \forall x (P(x) \Rightarrow Q(x)) \Rightarrow (\forall x P(x) \Rightarrow \forall x Q(x))$

c) $\Phi_3 \equiv \forall x (\varphi(x) \land \psi(x)) \Leftrightarrow (\forall x \varphi(x) \land \forall x \psi(x))$

d) $\Phi_4 \equiv \exists y \forall x (P(x, y) \Leftrightarrow P(x, x)) \Rightarrow \neg \forall x \exists y \forall z (P(z, y) \Leftrightarrow \neg P(z, x))$

Solution 1.1: A finished tableau in predicate logic is constructed analogously as in propositional logic. Additionally, a node of the form $T \exists x \varphi(x)$ (or $F \forall x \varphi(x)$) is reduced by adjoining $T \varphi(c)$ (or $F \varphi(c)$) to the end of every noncontradictory path involved. The letter $c$ represents a new constant that does not appear in any node on the expanded paths.

When nodes of the form $T \forall x \varphi(x)$ (or $F \exists x \varphi(x)$) are reduced, they should always be copied to the end of every noncontradictory path involved and are followed by $T \varphi(t)$ (or $F \varphi(t)$). The letter $t$ represents any ground term (term without variables). (The term is almost always constructed from function and constant symbols that occur on the particular paths.)
Finished tableaux for formulas $\Phi_1$ (left) and $\Phi_2$ (right):

\[
\begin{array}{l}
F(\forall x \varphi(x) \Rightarrow \neg(\exists x \neg \varphi(x))) \\
\quad \begin{array}{l}
T\forall x \varphi(x) \\
\quad F(\exists x \neg \varphi(x)) \\
\quad T\exists x \neg \varphi(x) \\
\quad T\neg \varphi(c) \text{ new } c \\
\quad F\varphi(c) \\
\quad T\forall x \varphi(x) \\
\quad T\varphi(c) \\
\quad \otimes
\end{array}
\end{array}
\quad \begin{array}{l}
F(\forall x P(x) \Rightarrow Q(x)) \Rightarrow (\forall x P(x) \Rightarrow \forall x Q(x)) \\
\quad \begin{array}{l}
T\forall x P(x) \Rightarrow Q(x)) \\
\quad F(\forall x P(x) \Rightarrow \forall x Q(x)) \\
\quad T\forall x P(x) \\
\quad F\forall x Q(x) \\
\quad FQ(c) \text{ new } c \\
\quad TP(c) \\
\quad T\forall x P(x) \Rightarrow Q(x)) \\
\quad T(P(c) \Rightarrow Q(c)) \\
\quad FP(c) \quad TQ(c) \\
\quad \otimes \quad \otimes
\end{array}
\end{array}
\]
Finished tableau for the formula $\Phi_3$:

$$F(\forall x(\varphi(x) \land \psi(x))) \Leftrightarrow (\forall x \varphi(x) \land \forall x \psi(x))$$

The last formula: analogically.

**Exercise 1.2:** Prove that the formula $\forall x P(x)$ is a logical consequence of the following formulas:

$$\forall x((Q(x) \lor R(x)) \Rightarrow \neg S(x))$$
$$\forall x((R(x) \Rightarrow \neg P(x)) \Rightarrow (Q(x) \land S(x)))$$

**Solution 1.2:** Tableau proofs of logical consequences in predicate logic are done in the same way as in propositional logic.
Exercise 1.3: Prove the following logical consequence using the tableau method. Assume that the following three statements hold:

- There exists a dragon (denote it $D/1$).
- Dragons sleep ($S/1$) or hunt ($L/1$).
- If a dragon is hungry ($H/1$), it cannot sleep.

Conclusion: *If a dragon is hungry, it hunts.*
Solution 1.3: Transformation into formulas:

Premises:

- $\exists x D(x)$
- $\forall x (D(x) \Rightarrow (S(x) \lor L(x)))$
- $\forall x ((D(x) \land H(x)) \Rightarrow \neg S(x))$

Conclusion:

- $\forall x ((D(x) \land H(x)) \Rightarrow L(x))$
Finished tableau for the logical consequence:

\[
F(\forall x((D(x) \land H(x)) \Rightarrow L(x)))
\]

\[
T(\exists x D(x))
\]

\[
T(\forall x(D(x) \Rightarrow (S(x) \lor L(x))))
\]

\[
T(\forall x((D(x) \land H(x)) \Rightarrow \neg S(x)))
\]

\[
F((D(c) \land H(c)) \Rightarrow L(c)) \quad \text{new } c
\]

\[
T(D(c) \land H(c))
\]

\[
F L(c)
\]

\[
T D(c)
\]

\[
T H(c)
\]

\[
T(\forall x(D(x) \Rightarrow (S(x) \lor L(x))))
\]

\[
T(D(c) \Rightarrow (S(c) \lor L(c)))
\]

\[
F D(c) \quad T D(c)
\]

\[
T S(c) \quad T L(c)
\]

\[
T(\forall x((D(x) \land H(x)) \Rightarrow \neg S(x)))
\]

\[
T((D(c) \land H(c)) \Rightarrow \neg S(c))
\]

\[
F(D(c) \land H(c)) \quad T \neg S(c)
\]

\[
F D(c) \quad F H(c) \quad F S(c)
\]