european social fund in the czech republic
EUROPEAN UNION
MINISTRY OF EDUCATION, YOUTH AND SPORTS
OP Education for Competitiveness
INVESTMENTS IN EDUCATION DEVELOPMENT

1 Tableaux in predicate logic

Exercise 1.1: Prove that the following formulas are tautologies. Use tableau method.

a) $\Phi_1 \equiv \forall x \varphi(x) \Rightarrow \neg(\exists x \neg\varphi(x))$

b) $\Phi_2 \equiv \forall x(P(x) \Rightarrow Q(x)) \Rightarrow (\forall x P(x) \Rightarrow \forall x Q(x))$

c) $\Phi_3 \equiv \forall x(\varphi(x) \wedge \psi(x)) \Leftrightarrow (\forall x \varphi(x) \wedge \forall x \psi(x))$

d) $\Phi_4 \equiv \exists y \forall x(P(x,y) \Leftrightarrow P(x,x)) \Rightarrow \neg\forall x \exists y \forall z(P(z,y) \Leftrightarrow \neg P(z,x))$

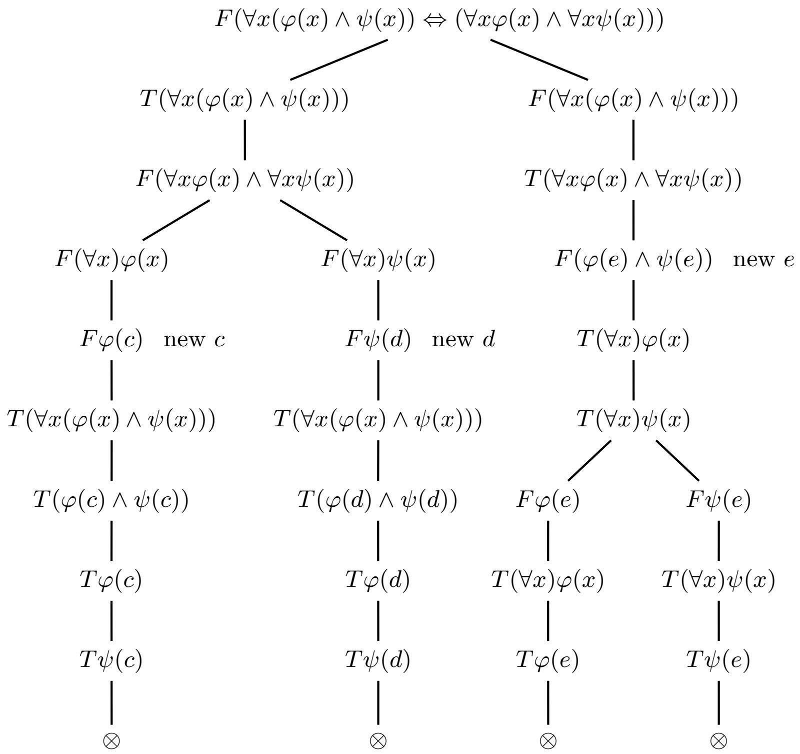
Solution 1.1: A finished tableau in predicate logic is constructed analogously as in propositional logic.
Additionally, a node of the form $T\exists x\varphi(x)$ (or $F\forall x\varphi(x)$) is reduced by adjoining $T\varphi(c)$ (or $F\varphi(c)$) to the end of every noncontradictory path involved. The letter c represents a new constant that does not appear in any node on the expanded paths.
When nodes of the form $T\forall x\varphi(x)$ (or $F\exists x\varphi(x)$) are reduced, they should always be copied to the end of every noncontradictory path involved and are followed by $T\varphi(t)$ (or $F\varphi(t)$). The letter t represents any ground term (term without variables). (The term is almost always constructed from function and constant symbols that occur on the particular paths.)

Finished tableaux for formulas Φ_1 (left) and Φ_2 (right):

$$
\begin{array}{c}
F(\forall x \varphi(x) \Rightarrow \neg(\exists x \neg\varphi(x))) \\
| \\
T\forall x \varphi(x) \\
| \\
F\neg(\exists x \neg\varphi(x)) \\
| \\
T\exists x \neg\varphi(x) \\
| \\
T\neg\varphi(c) \quad \text{new } c \\
| \\
F\varphi(c) \\
| \\
T\forall x \varphi(x) \\
| \\
T\varphi(c) \\
| \\
\otimes
\end{array}
$$

$$
\begin{array}{c}
F(\forall x(P(x) \Rightarrow Q(x)) \Rightarrow (\forall x P(x) \Rightarrow \forall x Q(x))) \\
| \\
T\forall x(P(x) \Rightarrow Q(x)) \\
| \\
F(\forall x P(x) \Rightarrow \forall x Q(x)) \\
| \\
T\forall x P(x) \\
| \\
F\forall x Q(x) \\
| \\
FQ(c) \quad \text{new } c \\
| \\
T\forall x P(x) \\
| \\
TP(c) \\
| \\
T\forall x(P(x) \Rightarrow Q(x)) \\
| \\
T(P(c) \Rightarrow Q(c)) \\
\diagup \qquad \diagdown \\
\begin{array}{cc}
FP(c) & TQ(c) \\
| & | \\
\otimes & \otimes
\end{array}
\end{array}
$$

Finished tableau for the formula Φ_3:

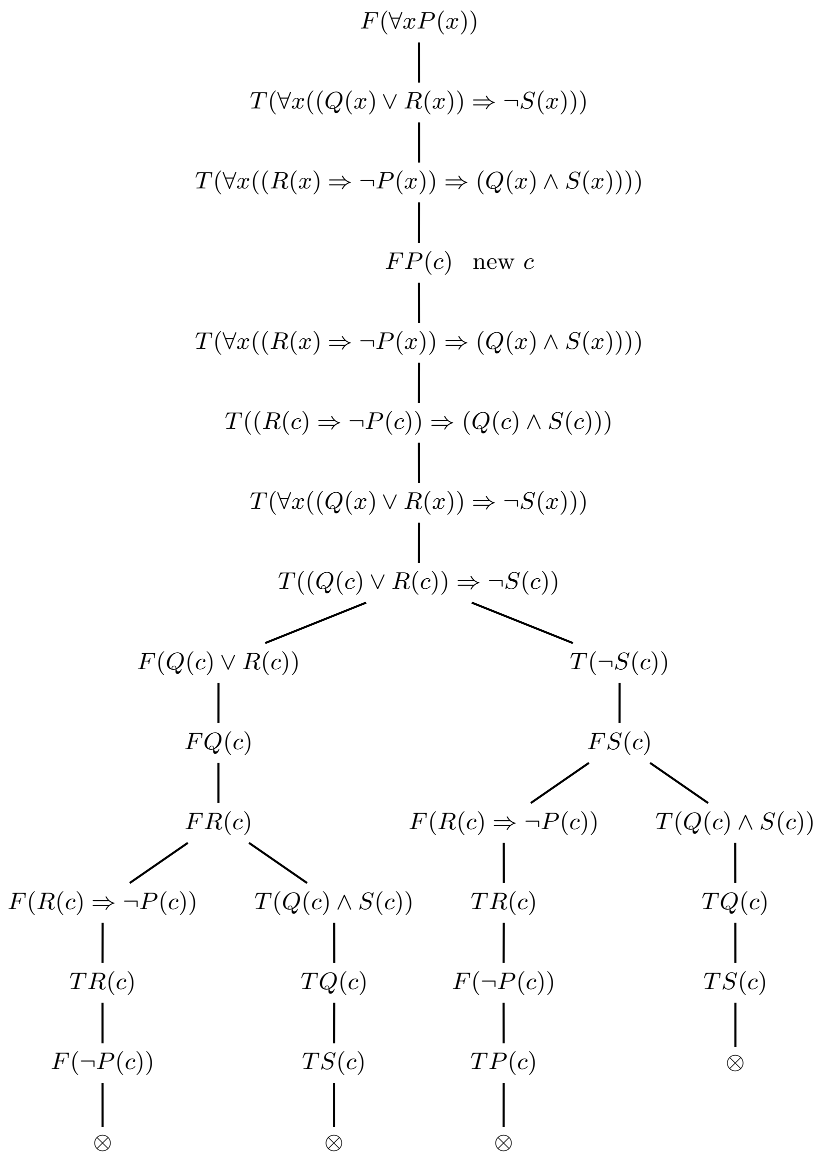


The last formula: analogically.

Exercise 1.2: Prove that the formula $\forall x P(x)$ is a logical consequence of the following formulas:

$$\forall x((Q(x) \lor R(x)) \Rightarrow \neg S(x))$$
$$\forall x((R(x) \Rightarrow \neg P(x)) \Rightarrow (Q(x) \land S(x)))$$

Solution 1.2: Tableau proofs of logical consequences in predicate logic are done in the same way as in propositional logic.



Exercise 1.3: Prove the following logical consequence using the tableau method. Assume that the following three statements hold:

- There exists a dragon (denote it $D/1$).
- Dragons sleep ($S/1$) or hunt ($L/1$).
- If a dragon is hungry ($H/1$), it cannot sleep.

Conclusion: *If a dragon is hungry, it hunts.*

Solution 1.3: Transformation into formulas:

$$
\begin{array}{ll}
\text{Premises:} & \exists x D(x) \\
& \forall x (D(x) \Rightarrow (S(x) \vee L(x))) \\
& \forall x ((D(x) \wedge H(x)) \Rightarrow \neg S(x)) \\
\text{Conclusion:} & \forall x ((D(x) \wedge H(x)) \Rightarrow L(x))
\end{array}
$$

Finished tableau for the logical consequence: