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EUROPEAN UNION
MINISTRY OF EDUCATION, YOUTH AND SPORTS
OP Education for Competitiveness
INVESTMENTS IN EDUCATION DEVELOPMENT

1 Modal logic

Exercise 1.1: Let us have the set of worlds $W = \{w_0, w_1, w_2\}$, an accessibility relation $S = \{(w_0, w_1), (w_0, w_2)\}$ and let $w_1 \Vdash p_2$. Which of the following statements hold?

a) $w_0 \Vdash \Diamond p_2$

b) $w_0 \Vdash \Box p_2$

c) $w_1 \Vdash \Box p_1$

d) $w_1 \Vdash \Box \neg p_1$

e) $w_0 \Vdash \Diamond \Box p_1$

f) $w_0 \Vdash \Box \Box p_1$

2 Tableaux in modal logic

Contradictory tableaux in modal logic are constructed in a similar way as in predicate logic. To prove that a formula φ is a tautology (i.e. it holds for all worlds of all Kripke frames over the used modal-logic language), it is necessary to construct a contradictory tableau with the root $Fw \Vdash \varphi$. In addition to predicate logic it is necessary to consider the world in which the formula should be true or false (it is captured in $w \Vdash$).

A path in a tableau is *contradictory* when it contains both $Tv \Vdash \varphi$ and $Fv \Vdash \varphi$ for the same world v and a formula φ.

Our modal logic language is supposed not to contain equivalence connectives and function symbols. When nodes of the form $Tv \Vdash \forall x\varphi(x)$ and $Fv \Vdash \exists x\varphi(x)$ are expanded, only constants are used (not ground terms). Only constants that belong to the particular world or to its predecessors can be used. When nodes $Tv \Vdash \exists x\varphi(x)$ or $Fv \Vdash \forall x\varphi(x)$ are expanded, a new constant (which is not present in any node of the tableau yet) should be used.

Nodes with toplevel operators $\Box$ and $\Diamond$ are reduced in the following way: when reducing $Tv \Vdash \Diamond\varphi$ or $Fv \Vdash \Box\varphi$ we first adjoin the node vSw to the end of the path (w is a new world that has not been used in the tableau yet). Then the node $Tw \Vdash \varphi$ or $Fw \Vdash \varphi$ is adjoined.
Nodes of the form $Tv \Vdash \Box\varphi$ or $Fv \Vdash \Diamond\varphi$ are expanded into $Tw \Vdash \varphi$ or $Fw \Vdash \varphi$ where w is an arbitrary world for which there is a node vSw on the expanded path. If it is not possible to get such a world on the path, we consider the nodes to be reduced.

Nodes of the form $Tv \Vdash \forall x\varphi(x)$, $Fv \Vdash \exists x\varphi(x)$, $Tv \Vdash \Box\varphi$ and $Fv \Vdash \Diamond\varphi$ should be always copied when reduced!

Exercise 2.1: Using tableaux prove that the following formulas are tautologies.

a) $\Phi_1 \equiv (\Box\forall x\varphi(x)) \Rightarrow (\forall x\Box\varphi(x))$

b) $\Phi_2 \equiv (\Box(\varphi \Rightarrow \psi)) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$

c) $\Phi_3 \equiv \neg\Diamond(\neg(\varphi \wedge \exists x\psi(x)) \wedge \exists x(\varphi \wedge \psi(x)))$, x is not free in the formula φ

d) $\Phi_4 \equiv \Diamond\exists x(\varphi(x) \Rightarrow \Box\psi) \Rightarrow \Diamond(\forall x\varphi(x) \Rightarrow \Box\psi)$, x is not free in the formula ψ

$$\begin{array}{c}
Fw \Vdash \forall x\Box\varphi(x) \Rightarrow \Box\forall x\varphi(x) \\
| \\
Tw \Vdash \forall x\Box\varphi(x) \\
| \\
Fw \Vdash \Box\forall x\varphi(x) \\
| \\
wSv \\
| \\
Fv \Vdash \forall x\varphi(x) \\
| \\
Fv \Vdash \varphi(c)\ new\ c \\
| \\
Tw \Vdash \forall x\Box\varphi(x)^{*} \\
| \\
Tw \Vdash \Box\varphi(c) \\
| \\
Tw \Vdash \Box\varphi(c) \\
| \\
Tv \Vdash \varphi(c) \\
| \\
\otimes
\end{array}$$

Figure 1:

Exercise 2.2: Consider the tableau with the root $Fw \Vdash \forall x\Box\varphi(x) \Rightarrow \Box\forall x\varphi(x)$ given in Figure 1. Decide whether the tableau is correct or not. Explain your decision.

Exercise 2.3: Prove the following logical consequences:

a) $\{\varphi\} \models \Box\varphi$

b) $\{\forall x\varphi(x)\} \models \Box\forall x\varphi(x)$

c) $\{\forall x \varphi(x)\} \models \forall x \Box \varphi(x)$

d) $\{\varphi \Rightarrow \Box \varphi\} \models \Box \varphi \Rightarrow \Box\Box \varphi$