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EUROPEAN UNION



MINISTRY OF EDUCATION,
YOUTH AND SPORTS



OP Education
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INVESTMENTS IN EDUCATION DEVELOPMENT

1 Modal logic

Exercise 1.1: Let us have the set of worlds $W = \{w_0, w_1, w_2\}$, an accessibility relation $S = \{(w_0, w_1), (w_0, w_2)\}$ and let $w_1 \Vdash p_2$. Which of the following statements hold?

- a) $w_0 \Vdash \Diamond p_2$
- b) $w_0 \Vdash \Box p_2$
- c) $w_1 \Vdash \Box p_1$
- d) $w_1 \Vdash \Box \neg p_1$
- e) $w_0 \Vdash \Diamond \Box p_1$
- f) $w_0 \Vdash \Box \Box p_1$

2 Tableaux in modal logic

Contradictory tableaux in modal logic are constructed in a similar way as in predicate logic. To prove that a formula φ is a tautology (i.e. it holds for all worlds of all Kripke frames over the used modal-logic language), it is necessary to construct a contradictory tableau with the root $Fw \Vdash \varphi$. In addition to predicate logic it is necessary to consider the world in which the formula should be true or false (it is captured in $w \Vdash$).

A path in a tableau is *contradictory* when it contains both $Tv \Vdash \varphi$ and $Fv \Vdash \varphi$ for the same world v and a formula φ .

Our modal logic language is supposed not to contain equivalence connectives and function symbols. When nodes of the form $Tv \Vdash \forall x\varphi(x)$ and $Fv \Vdash \exists x\varphi(x)$ are expanded, only constants are used (not ground terms). Only constants that belong to the particular world or to its predecessors can be used. When nodes $Tv \Vdash \exists x\varphi(x)$ or $Fv \Vdash \forall x\varphi(x)$ are expanded, a new constant (which is not present in any node of the tableau yet) should be used.

Nodes with toplevel operators \Box and \Diamond are reduced in the following way: when reducing $Tv \Vdash \Diamond\varphi$ or $Fv \Vdash \Box\varphi$ we first adjoin the node vSw to the end of the path (w is a new world that has not been used in the tableau yet). Then the node $Tw \Vdash \varphi$ or $Fw \Vdash \varphi$ is adjoined.

Nodes of the form $Tv \Vdash \Box\varphi$ or $Fv \Vdash \Diamond\varphi$ are expanded into $Tw \Vdash \varphi$ or $Fw \Vdash \varphi$ where w is an arbitrary world for which there is a node vSw on the expanded path. If it is not possible to get such a world on the path, we consider the nodes to be reduced.

Nodes of the form $Tv \Vdash \forall x\varphi(x)$, $Fv \Vdash \exists x\varphi(x)$, $Tv \Vdash \Box\varphi$ and $Fv \Vdash \Diamond\varphi$ should be always copied when reduced!

Exercise 2.1: Using tableaux prove that the following formulas are tautologies.

- a) $\Phi_1 \equiv (\Box \forall x \varphi(x)) \Rightarrow (\forall x \Box \varphi(x))$
- b) $\Phi_2 \equiv (\Box(\varphi \Rightarrow \psi)) \Rightarrow (\Box \varphi \Rightarrow \Box \psi)$
- c) $\Phi_3 \equiv \neg \Diamond(\neg(\varphi \wedge \exists x \psi(x)) \wedge \exists x(\varphi \wedge \psi(x)))$, x is not free in the formula φ
- d) $\Phi_4 \equiv \Diamond \exists x(\varphi(x) \Rightarrow \Box \psi) \Rightarrow \Diamond(\forall x \varphi(x) \Rightarrow \Box \psi)$, x is not free in the formula ψ

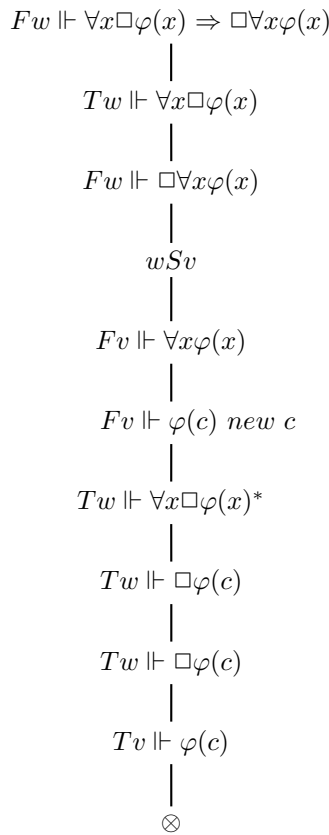


Figure 1:

Exercise 2.2: Consider the tableau with the root $Fw \Vdash \forall x \Box \varphi(x) \Rightarrow \Box \forall x \varphi(x)$ given in Figure 1. Decide whether the tableau is correct or not. Explain your decision.

Exercise 2.3: Prove the following logical consequences:

- a) $\{\varphi\} \models \Box \varphi$
- b) $\{\forall x \varphi(x)\} \models \Box \forall x \varphi(x)$

c) $\{\forall x\varphi(x)\} \models \forall x\Box\varphi(x)$

d) $\{\varphi \Rightarrow \Box\varphi\} \models \Box\varphi \Rightarrow \Box\Box\varphi$