european social fund in the czech republic
EUROPEAN UNION
MINISTRY OF EDUCATION, YOUTH AND SPORTS
OP Education for Competitiveness
INVESTMENTS IN EDUCATION DEVELOPMENT

1 Modal logic

Exercise 1.1: Let us have the set of worlds $W = \{w_0, w_1, w_2\}$, an accessibility relation $S = \{(w_0, w_1), (w_0, w_2)\}$ and let $w_1 \Vdash p_2$. Which of the following statements hold?

a) $w_0 \Vdash \Diamond p_2$

b) $w_0 \Vdash \Box p_2$

c) $w_1 \Vdash \Box p_1$

d) $w_1 \Vdash \Box \neg p_1$

e) $w_0 \Vdash \Diamond \Box p_1$

f) $w_0 \Vdash \Box \Box p_1$

Solution 1.1: The only statement which does not hold is b).

2 Tableaux in modal logic

Contradictory tableaux in modal logic are constructed in a similar way as in predicate logic. To prove that a formula φ is a tautology (i.e. it holds for all worlds of all Kripke frames over the used modal-logic language), it is necessary to construct a contradictory tableau with the root $Fw \Vdash \varphi$. In addition to predicate logic it is necessary to consider the world in which the formula should be true or false (it is captured in $w \Vdash$).

A path in a tableau is *contradictory* when it contains both $Tv \Vdash \varphi$ and $Fv \Vdash \varphi$ for the same world v and a formula φ.

Our modal logic language is supposed not to contain equivalence connectives and function symbols. When nodes of the form $Tv \Vdash \forall x \varphi(x)$ and $Fv \Vdash \exists x \varphi(x)$ are expanded, only constants are used (not ground terms). Only constants that belong to the particular world or to its predecessors can be used. When nodes $Tv \Vdash \exists x \varphi(x)$ or $Fv \Vdash \forall x \varphi(x)$ are expanded, a new constant (which is not present in any node of the tableau yet) should be used.

Nodes with toplevel operators $\Box$ and $\Diamond$ are reduced in the following way: when reducing $Tv \Vdash \Diamond \varphi$ or $Fv \Vdash \Box \varphi$ we first adjoin the node vSw to the end of the path (w is a new world that has not been used in the tableau yet). Then the node $Tw \Vdash \varphi$ or $Fw \Vdash \varphi$ is adjoined.
Nodes of the form $Tv \Vdash \Box \varphi$ or $Fv \Vdash \Diamond \varphi$ are expanded into $Tw \Vdash \varphi$ or $Fw \Vdash \varphi$ where w is an arbitrary world for which there is a node vSw on the expanded path. If it is not possible to get such a world on the path, we consider the nodes to be reduced.

Nodes of the form $Tv \Vdash \forall x\varphi(x)$, $Fv \Vdash \exists x\varphi(x)$, $Tv \Vdash \Box\varphi$ and $Fv \Vdash \Diamond\varphi$ should be always copied when reduced!

Exercise 2.1: Using tableaux prove that the following formulas are tautologies.

a) $\Phi_1 \equiv (\Box\forall x\varphi(x)) \Rightarrow (\forall x\Box\varphi(x))$

b) $\Phi_2 \equiv (\Box(\varphi \Rightarrow \psi)) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$

c) $\Phi_3 \equiv \neg\Diamond(\neg(\varphi \wedge \exists x\psi(x)) \wedge \exists x(\varphi \wedge \psi(x)))$, x is not free in the formula φ

d) $\Phi_4 \equiv \Diamond\exists x(\varphi(x) \Rightarrow \Box\psi) \Rightarrow \Diamond(\forall x\varphi(x) \Rightarrow \Box\psi)$, x is not free in the formula ψ

Solution 2.1: See Figures 1 and 2.

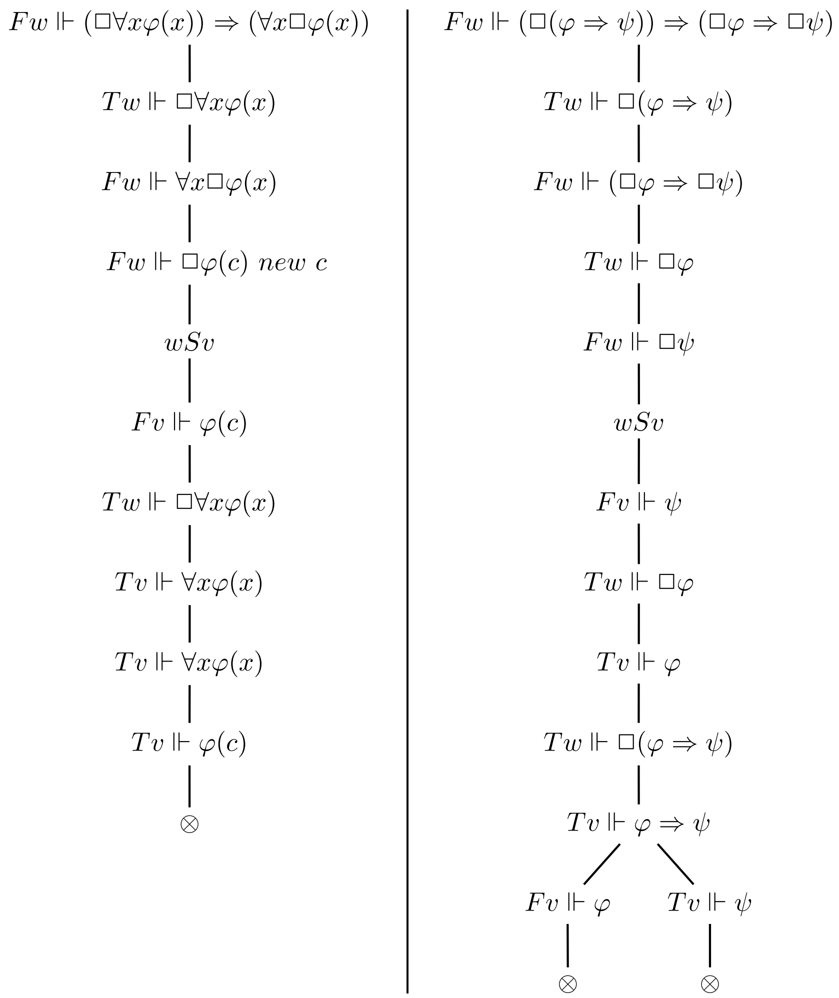

Figure 1: Finished contradictory tableaux for Φ_1 (left) and Φ_2 (right).

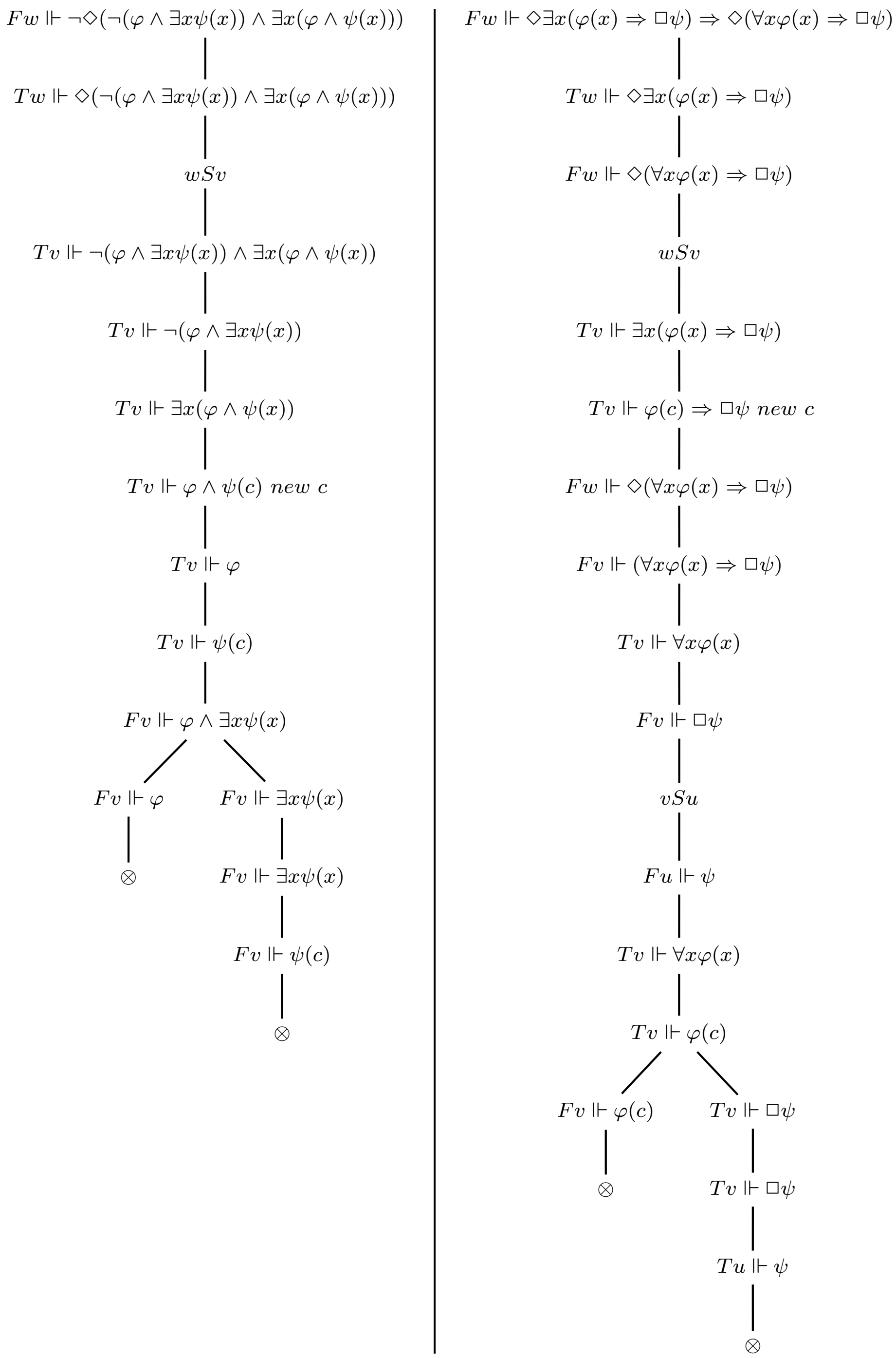

Figure 2: Finished contradictory tableaux for Φ_3 (left) and Φ_4 (right).

$$Fw \Vdash \forall x \Box \varphi(x) \Rightarrow \Box \forall x \varphi(x)$$
$$|$$
$$Tw \Vdash \forall x \Box \varphi(x)$$
$$|$$
$$Fw \Vdash \Box \forall x \varphi(x)$$
$$|$$
$$wSv$$
$$|$$
$$Fv \Vdash \forall x \varphi(x)$$
$$|$$
$$Fv \Vdash \varphi(c) \ \mathit{new}\ c$$
$$|$$
$$Tw \Vdash \forall x \Box \varphi(x)^*$$
$$|$$
$$Tw \Vdash \Box \varphi(c)$$
$$|$$
$$Tw \Vdash \Box \varphi(c)$$
$$|$$
$$Tv \Vdash \varphi(c)$$
$$|$$
$$\otimes$$

Figure 3:

Exercise 2.2: Consider the tableau with the root $Fw \Vdash \forall x \Box \varphi(x) \Rightarrow \Box \forall x \varphi(x)$ given in Figure 3. Decide whether the tableau is correct or not. Explain your decision.

Solution 2.2: The tableau is not correct (in our modal logic) and it does not prove that the formula is a tautology.

When we expand the node $(^*)$ it is not possible to replace the variable x with the constant c. The constant c appeared in the world v, however, we are in the world w in the node $(^*)$. The expansion would be correct if the world w was a successor of world v, not its predecessor.

Exercise 2.3: Prove the following logical consequences:

a) $\{\varphi\} \models \Box \varphi$

b) $\{\forall x \varphi(x)\} \models \Box \forall x \varphi(x)$

c) $\{\forall x \varphi(x)\} \models \forall x \Box \varphi(x)$

d) $\{\varphi \Rightarrow \Box\varphi\} \models \Box\varphi \Rightarrow \Box\Box\varphi$

Solution 2.3: The tableau for the logical consequence $S \models \varphi$ starts with $Fv \Vdash \varphi$ and at any time we can add to the end of any path P a node of the form $Tw \Vdash \alpha$, where w is any world that appears somewhere on the path P and $\alpha \in S$.

Finished contradictory tableaux for the consequences (a),(b) and (c) are in the picture 4. The tableau for the consequence (d) is left as an exercise.

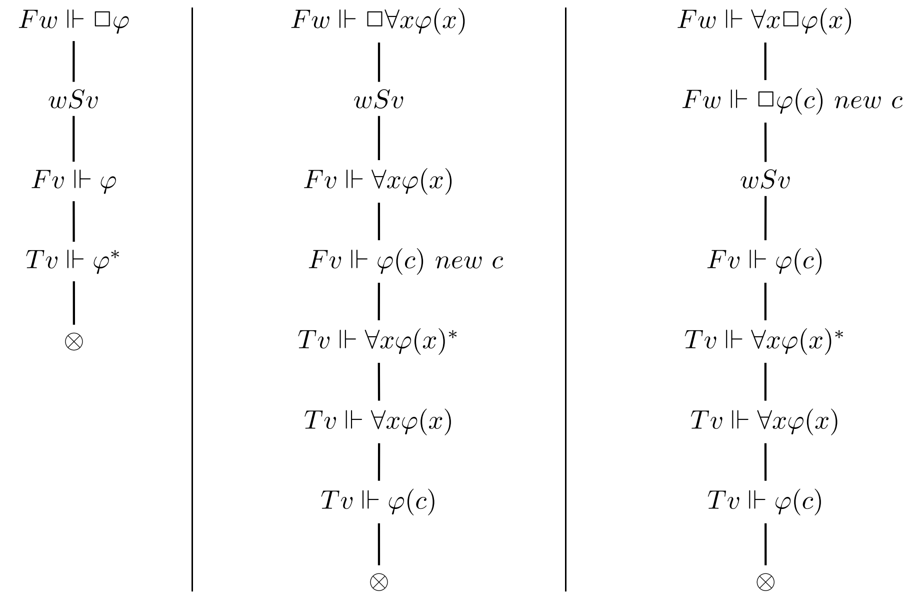

Figure 4: Finished contradictory tableaux for the logical consequences (a),(b) and (c). Nodes that correspond to the addition of premises are marked with an asterisk.