INVESTMENTS IN EDUCATION DEVELOPMENT

1 Transformation into CNF

Exercise 1.1: Convert the following formulas into CNF using truth tables.

a) $(p \Leftrightarrow q) \Rightarrow (\neg p \wedge r)$

b) $(p \Rightarrow q) \Rightarrow r$

c) $p \Leftrightarrow q$

Solution 1.1: Formula: $(p \Leftrightarrow q) \Rightarrow (\neg p \wedge r)$

Truth table:

p	q	r	$p \Leftrightarrow q$	$\neg p \wedge r$	$(p \Leftrightarrow q) \Rightarrow (\neg p \wedge r)$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	0	0	1
0	1	1	0	1	1
1	0	0	0	0	1
1	0	1	0	0	1
1	1	0	1	0	0
1	1	1	1	0	0

Solution: (full) conjunctive normal form (FCNF) of the formula is
$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

Other formulas: analogically.

Exercise 1.2: Convert the following formulas into CNF using equivalent transformations.

a) $(p \Rightarrow q) \Leftrightarrow (p \Rightarrow r)$

b) $(p \vee q) \wedge (p \wedge r)$

c) $(p \Rightarrow q) \Rightarrow r$

d) $p \Leftrightarrow q$

Solution 1.2: Formula: $(p \Rightarrow q) \Leftrightarrow (p \Rightarrow r)$

Transformations:

$$((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \wedge ((p \Rightarrow r) \Rightarrow (p \Rightarrow q))$$
$$(\neg(p \Rightarrow q) \vee (p \Rightarrow r)) \wedge (\neg(p \Rightarrow r) \vee (p \Rightarrow q))$$
$$((p \wedge \neg q) \vee (\neg p \vee r)) \wedge ((p \wedge \neg r) \vee (\neg p \vee q))$$
$$(((p \vee \neg p) \wedge (\neg q \vee \neg p)) \vee r) \wedge (((p \vee \neg p) \wedge (\neg r \vee \neg p)) \vee q)$$
$$(\neg q \vee \neg p \vee r) \wedge (\neg r \vee \neg p \vee q)$$

Solution: $(\neg q \vee \neg p \vee r) \wedge (\neg r \vee \neg p \vee q)$

Other formulas: analogically. Notice that the formula $(p \vee q) \wedge (p \wedge r)$ is in CNF. It can be read as $(p \vee q) \wedge p \wedge r$.

2 Resolution in propositional logic

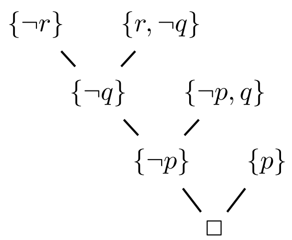
Exercise 2.1: Prove the following logical consequence using resolution:
$\neg p \vee q, \neg r \Rightarrow \neg q \models p \Rightarrow r$

Solution 2.1:
Negation of the conclusion:
$p \wedge \neg r$

Transfer of the negation to the set of premises:
$\{\neg p \vee q, \neg r \Rightarrow \neg q, p \wedge \neg r\}$

Conversion into CNF, set notation:
$\{\{\neg p, q\}, \{r, \neg q\}, \{p\}, \{\neg r\}\}$

Resolution refutation:



Conclusion: the logical consequence holds.

3 Transformation into PNF, Skolemization

Exercise 3.1: Transform the following formulas into (conjunctive) PNF:

a) $\exists z(\forall y(\exists x P(x, y) \Rightarrow Q(y, z)) \wedge \exists y(\forall x R(x, y) \vee Q(z, y)))$

b) $\forall x R(x) \Rightarrow \forall y P(y)$

c) $\forall x \exists y P(x, y) \vee \exists x \forall z R(f(x))$

d) $\forall y \exists x R(x, y) \Leftrightarrow \forall x \forall y P(x, y)$

e) $(\forall x \exists y Q(x, y) \vee \exists x \forall y P(x, y)) \wedge \neg \exists x \exists y P(x, y)$

Solution 3.1:

Formula: $\exists z(\forall y(\exists x P(x, y) \Rightarrow Q(y, z)) \wedge \exists y(\forall x R(x, y) \vee Q(z, y)))$

Transformations:

$$
\begin{aligned}
& \exists z(\forall y(\exists x P(x, y) \Rightarrow Q(y, z)) \wedge \exists y(\forall x R(x, y) \vee Q(z, y))) \Leftrightarrow \\
\Leftrightarrow\ & \exists z(\forall y(\neg \exists x P(x, y) \vee Q(y, z)) \wedge \exists y(\forall x R(x, y) \vee Q(z, y))) \Leftrightarrow \\
\Leftrightarrow\ & \exists z(\forall y(\neg \exists x P(x, y) \vee Q(y, z)) \wedge \exists y_1(\forall x_1 R(x_1, y_1) \vee Q(z, y_1))) \Leftrightarrow \\
\Leftrightarrow\ & \exists z(\forall y(\forall x \neg P(x, y) \vee Q(y, z)) \wedge \exists y_1(\forall x_1 R(x_1, y_1) \vee Q(z, y_1))) \Leftrightarrow \\
\Leftrightarrow\ & \exists z(\forall y \forall x(\neg P(x, y) \vee Q(y, z)) \wedge \exists y_1 \forall x_1(R(x_1, y_1) \vee Q(z, y_1))) \Leftrightarrow \\
\Leftrightarrow\ & \exists z \forall y \forall x((\neg P(x, y) \vee Q(y, z)) \wedge \exists y_1 \forall x_1(R(x_1, y_1) \vee Q(z, y_1))) \Leftrightarrow \\
\Leftrightarrow\ & \exists z \forall y \forall x \exists y_1 \forall x_1((\neg P(x, y) \vee Q(y, z)) \wedge (R(x_1, y_1) \vee Q(z, y_1)))
\end{aligned}
$$

Solution: $\exists z \forall y \forall x \exists y_1 \forall x_1((\neg P(x, y) \vee Q(y, z)) \wedge (R(x_1, y_1) \vee Q(z, y_1)))$

Other formulas: analogically. Notice that the formula and its PNF are equivalent. A PNF of a formula is *not* necessarily unique.

Exercise 3.2: Convert the following formulas into a Skolem normal form:

a) $\exists z \forall y \forall x \exists y_1 \forall x_1((\neg P(x, y) \vee Q(y, z)) \wedge (R(x_1, y_1) \vee Q(z, y_1)))$

b) $\exists x \forall y P(a, y, x)$

c) $\forall y \exists x P(f(x), y, x)$

d) $\exists x R(x) \Rightarrow \forall x P(x)$

e) $(\forall x \exists y Q(x, y) \vee \exists x \forall y P(x, y)) \wedge \neg \exists x \exists y P(x, y)$

Solution 3.2:

Formula: $\exists z \forall y \forall x \exists y_1 \forall x_1((\neg P(x, y) \vee Q(y, z)) \wedge (R(x_1, y_1) \vee Q(z, y_1)))$

Transformations:

$$
\begin{aligned}
& \exists z \forall y \forall x \exists y_1 \forall x_1((\neg P(x, y) \vee Q(y, z)) \wedge (R(x_1, y_1) \vee Q(z, y_1))) \\
& \forall y \forall x \exists y_1 \forall x_1((\neg P(x, y) \vee Q(y, c)) \wedge (R(x_1, y_1) \vee Q(c, y_1))) \\
& \forall y \forall x \forall x_1((\neg P(x, y) \vee Q(y, c)) \wedge (R(x_1, f(y, x)) \vee Q(c, f(y, x))))
\end{aligned}
$$

Solution: $\forall y \forall x \forall x_1((\neg P(x,y) \lor Q(y,c)) \land (R(x_1, f(y,x)) \lor Q(c, f(y,x))))$

Other formulas: analogically. First, it is necessary to convert the formulas into PNF. Notice that the formula and its SNF are *not* necessarily equivalent. However, they are *equisatisfiable*.

4 Unification

Exercise 4.1: For the following sets of literals find the most general unifier (mgu) or explain why it does not exist. Letters a, b, c represent constants.

a) $\{P(x, f(y), z), P(g(a), f(w), u), P(v, f(b), c)\}$

b) $\{P(x, f(x)), P(f(x), x)\}$

c) $\{P(a, x), P(a, y)\}$

d) $\{P(a, x), P(b, x)\}$

e) $\{P(x), R(x)\}$

f) $\{Q(h(x, y), w), Q(h(g(v), a), f(v)), Q(h(g(v), a), f(b))\}$

Solution 4.1: We find disagreements of set elements and (if possible) apply substitutions that reduce disagreements until the set is a singleton (contains one element).

Solution: mgu

a) $\{x/g(a), v/g(a), y/b, w/b, z/c, u/c\}$

b) does not exist

c) $\{x/y\}$

d) does not exist

e) does not exist

f) $\{x/g(b), y/a, w/f(b), v/b\}$