1 Inductive inference in propositional logic

We use Prolog notation for both the data and the learned hypotheses. For example, let us have the following data with three attributes classified into two classes `true/false`:

`Size` ∈ {`small, medium, large`},
`Color` ∈ {`red, blue, green`},
`Shape` ∈ {`square, circle, triangle`}

small	red	triangle	true
small	green	triangle	true
large	red	triangle	false
small	blue	circle	false

Representation of positive examples:
```
p(small,red,triangle).
p(small,green,triangle).
```
and similarly for negative examples:
```
p(large,red,triangle).
p(small,blue,circle).
```

Our goal is to find hypotheses that cover all positive examples and don't cover any negative example. The learned hypotheses are supposed to be in the form of clauses (Prolog rules):
`p(X,Y,Z):-`
<conditions for the attributes/variables `X,Y,Z` in form `Variable=value`>

The most general hypothesis (covers all examples) is `p(X,Y,Z).`
The most specific hypothesis (covers no example) is `p(X,Y,Z):- false.`

For a clause we can get its

- minimal proper generalization by removing a condition from the body of the clause
- minimal proper specialization by adding a condition to the body of the clause

Example of a specialization:
`p(X,Y,Z):- X=small.`
(this is the minimal (proper) specialization of `p(X,Y,Z)`)

Exercise 1.1: For the language
`Size` $\in$ {`small, medium, large`},
`Color` $\in$ {`red, blue, green`},
`Shape` $\in$ {`square, circle, triangle`}
find a proper specialization (all proper specializations) of clauses

a) `p(Size,Color,Shape) :- Size=large, Color=red.`

b) `p(Size,Color,Shape) :- Color=red.`

c) `p(Size,Color,Shape) :- Size=large, Color=red, Shape=circle.`

d) `p(Size,Color,Shape).`

e) Find all proper specializations of the clause `p(Size,Color,Shape) :- Color=red.` that cover `p(large,red,square).`

f) Find out whether (and how) the formulas from items a)–d) are in the generalization/specialization relation.

Solution 1.1:

a) `p(Size,Color,Shape) :- Size=large, Color=red, A.`, where `A` is an arbitrary condition of the form `Shape=value`. Another proper specialization (which is usually ommited) is `"false"`.

b) `p(Size,Color,Shape) :- Color=red, A, B.`, where `A` is an arbitrary condition of the form `Size=value`, `B` is an arbitrary condition of the form `Shape=value`; the order of `A` and `B` is not important here. We can get further specializations either omitting `A`, or `B`. Another proper specialization is `"false"`.

c) Apart from `"false"` there is no other proper specialization.

d) analogously as in b), we arbitrarily apply conditions to all of the three attributes

e) `p(Size,Color,Shape) :- Color=red, Size=large.`
`p(Size,Color,Shape) :- Color=red, Shape=square.`
`p(Size,Color,Shape) :- Color=red, Size=large, Shape=square.`

f) for example a) is a specialization of b) etc.

Exercise 1.2: For the language
$0 <$ `Size` ≤ 1,
`Color` $\in$ {`red, blue`},
`Square` $\in$ {`yes, no`}
describe the space of (conjunctive) hypotheses. First, you can do this just for the language without the continuous attribute `Size`.

Then find some proper specializations of clauses

a) `p(Size,Color,Square) :- Square=yes, Color=red.`

b) `p(Size,Color,Square) :- Color=red.`

c) `p(Size,Color,Square) :- Size < 0.1, Color=red, Square=no.`

d) `p(Size,Color,Square).`

Solution 1.2: The space of conjunctive hypotheses is a lattice. The least element is `"false"`, the greatest element is the most general hypothesis. The elements between them represent various specializations. Two elements are connected with an edge, if they are in the generalization/specialization relation.
For the continuous domain of an attribut we divide the interval into two subintervals. Subintervals can be further divided in the same way.

Specializations are constructed analogously to the former example. For the continuous attribute `Size` we divide the interval arbitrarily (typically with respect to the data), each time into two subintervals. Then, a specialization has to respect this selected division (discretization).

For the discretization
`(0,1>` divided into `(0,0.3>` `(0.3,1>` and
`(0.3,1>` further divided into `(0.3,0.7>` `(0.7,1>`
are examples of possible specializations for the subtask b) e.g.:

```
p(Size,Color,Square) :- Color=red, Size < 0.2.
p(Size,Color,Square) :- Color=red, Size > 0.5, Size < 0.8.
p(Size,Color,Square) :- Color=red, Size > 0.8, Size < 0.9.
```

while this clause is not supposed to be a specialization wrt. the discretization:

```
p(Size,Color,Square) :- Color=red, Size < 0.5.
```

Exercise 1.3: For the following data find all its disjunctive concepts:

small	red	triangle	true
small	green	triangle	true
large	red	triangle	true
small	blue	circle	false
small	blue	triangle	false

Solution 1.3: A disjunctive concept is a set of clauses (while a conjunctive hypothesis consists of one clause only).

One of possible disjunctive concepts:

```
p(Size,Color,Shape) :- Color=red.
p(Size,Color,Shape) :- Color=green.
```

We can get other disjunctive concepts using various specializations (or splitting) of its clauses. However, it is necessary to ensure that all positive examples are still covered:

```
p(Size,Color,Shape) :- Color=red, Shape=triangle.
p(Size,Color,Shape) :- Color=green.

p(Size,Color,Shape) :- Color=red, Shape=triangle.
p(Size,Color,Shape) :- Color=green, Size=small.
```

```
p(Size,Color,Shape) :- Color=red.
p(Size,Color,Shape) :- Color=green, Size=small, Shape=triangle.

...
```